THE KLEIN-GORDON EQUATION ON $\mathbb{Z}^2$ AND THE QUANTUM HARMONIC LATTICE

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ABSTRACT. The discrete Klein-Gordon equation on a two-dimensional square lattice satisfies an $\ell^1 \mapsto \ell^\infty$ dispersive bound with polynomial decay rate $|t|^{-3/4}$. We determine the shape of the light cone for any choice of the mass parameter and relative propagation speeds along the two coordinate axes. Fundamental solutions experience the least dispersion along four caustic lines interior to the light cone rather than along its boundary, and decay exponentially to arbitrary order outside the light cone. The overall geometry of the propagation pattern and its associated dispersive bounds are independent of the particular choice of parameters. In particular there is no bifurcation of the number or type of caustics that are present.

The discrete Klein-Gordon equation is a classical analogue of the quantum harmonic lattice. In the quantum setting, commutators of time-shifted observables experience the same decay rates as the corresponding Klein-Gordon solutions, which depend in turn on the relative location of the observables' support sets.

1. INTRODUCTION

The wave equation $u_{tt} - \Delta u = 0$ on $\mathbb{R}^{2+1}$ is explicitly solved via Poisson's formula, in which initial data $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ determines the unique solution

$$u(x,t) = \frac{\operatorname{sign}(t)}{2\pi} \int_{|y-x|<|t|} \frac{h(y) + \frac{1}{t}(g(y) + \nabla g(y)\cdot(y-x))}{\sqrt{t^2 - |y-x|^2}}\, dy$$

at any $t \neq 0$. More generally the Klein-Gordon equation $u_{tt} - \Delta u + m^2 u = 0$ with the same initial data has the solution

$$u(x,t) = \frac{\operatorname{sign}(t)}{2\pi} \int_{|y-x|<|t|} \Big(h(y) + \frac{1}{t}(g(y) + \nabla g(y)\cdot(y-x))\Big) \frac{\cos(m\sqrt{t^2 - |y-x|^2}\,)}{\sqrt{t^2 - |y-x|^2}}\, dy.$$

When $m = 0$ the two equations coincide. It is clear in these formulas that the propagator kernel is radially symmetric, and that all information from the initial data travels at finite speed. Both equations also possess a well-known dispersive property that $|u(x,t)| \leq C|t|^{-1/2}$ provided the initial data are sufficiently smooth and decay at infinity.

This paper investigates the propagation patterns and dispersive bounds for solutions to a family of discrete Klein-Gordon equations on $\mathbb{Z}^2 \times \mathbb{R}^1$,

$$\begin{cases} u_{tt}(x,t) \underbrace{-\sum_{j=1}^2 \lambda_j\big(u(x+e_j,t) + u(x-e_j,t) - 2u(x,t)\big) + \omega^2 u(x,t)}_{Hu} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases} \tag{1.1}$$

with $\omega, \lambda_1, \lambda_2 > 0$ fixed parameters. The conserved energy is given by

$$E(t) = \frac{1}{2}\sum_{x\in\mathbb{Z}^2}\Big(u_t^2(x,t) + \omega^2 u^2(x,t) + \sum_{j=1}^2 \lambda_j (u(x+e_j,t) - u(x,t))^2\Big) \tag{1.2}$$

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and (by analogy with the continuous setting) the values of λ_j suggest propagation speeds of $\sqrt{\lambda_j}$ along their respective coordinate directions. We have chosen the letter ω for the mass parameter in order to highlight connections between (1.1) and the quantum harmonic lattice system.

The one-dimensional wave equation provides a certain degree of inspiration; when $\lambda = 1$ its fundamental solution is expressed in terms of the Bessel functions $J_{|y-x|}(t)$ (see [11]). There are three main asymptotic regimes. For $|t| \gg |y-x|$ there is oscillation with amplitude $|t|^{-1/2}$. When $|t| \ll |y-x|$ the propagator is nonzero (hence there is some rapid transfer of information) with exponential decay at spatial infinity on the order of $((2|y-x|)^{-1}et)^{|y-x|}$. For $|y-x| = |t| + O(t^{1/3})$ the propagator kernel reaches its maximum size of approximately $|t|^{-1/3}$. This bound is most easily obtained by applying van der Corput's lemma to the Fourier representation of $J_{|y-x|}(t)$.

Unfortunately the discrete wave and Klein-Gordon equations in higher dimensions do not separate variables as does the Schrödinger equation. The fundamental solution can still be determined as a superposition of plane waves, with size bounds in the different regimes resulting from stationary phase principles. The two and three-dimensional isotropic wave equations ($\lambda_j = 1$, $\omega = 0$) were analyzed by Schultz [16], with a curious set of outcomes. In addition to the expected wavefront expanding radially at $|y-x| \sim |t|$, there is a secondary region of reduced dispersion traveling at somewhat lower speed. In two dimensions the region lies along an astroid-shaped curve with diameter $\sqrt{2}\,|t|$; in three dimensions the region follows a cusped and pointed surface of a similar nature. Surprisingly, some global dispersive bounds are dominated by behavior when $|y-x|$ belongs to the secondary set, even though this occurs well inside the overall propagation pattern.

In the two-dimensional discrete Klein-Gordon equation, each plane wave $u_k(x) := e^{ik\cdot x}$ satisfies $Hu_k = \gamma^2(k)u_k$, with $\gamma^2(k) = \omega^2 + \sum_j 2\lambda_j(1-\cos k_j)$ and k ranging over the fundamental domain $[-\pi,\pi]^2$. The solution of (1.1) is given formally by

$$u(x,t) = \cos(t\sqrt{H})g + \frac{\sin(t\sqrt{H})}{\sqrt{H}}h \qquad \text{and} \qquad u_t(x,t) = \sqrt{H}\,\sin(t\sqrt{H})g + \cos(t\sqrt{H})h.$$

The operators involved act on a plane wave u_k by multiplication by $\cos(t\gamma(k))$, $\frac{\sin(t\gamma(k))}{\gamma(k)}$, and $\gamma(k)\sin(t\gamma(k))$, hence the fundamental solutions of (1.1) in physical space will be the inverse Fourier transform of those three functions. For all practical purposes these are oscillatory integrals over the torus $k \in [-\pi,\pi]^2$ of the form given in (2.2), whose asymptotic behavior is governed by critical points of the phase function $t\gamma(k) \pm x\cdot k$.

Three distinct regimes again emerge: critical points are absent for $x \gg t$ and exponential decay is observed by following the analytic continuation of $\gamma(k)$ into $\{[-\pi,\pi] + i\mathbb{R}\}^2$. Quantitative exponential bounds are given in Theorems 2.6 and 2.7. For generic values of (x,t) inside the "light cone" (i.e. $x = t\nabla\gamma(k)$ for some k), stationary phase arguments lead to a bound of $|t|^{-1}$. Along the boundary of the light cone, and within the secondary region introduced above, degenerate stationary phase estimates yield polynomial time decay with a fractionally smaller exponent. For fixed $t \neq 0$ there is a global bound of order $|t|^{-3/4}$ with maxima occuring near the four cusps of the astroid curve. This is a faster rate of decay than the discrete Schrödinger equation on $\mathbb{Z}^2$, where separation of variables leads to a $|t|^{-2/3}$ bound instead. Further details about the structure of the Klein-Gordon propagators are summarized in Theorem 2.3 and its corollaries.

The problem of generalizing van der Corput's lemma to two and higher dimensions has a long history in harmonic analysis. Since the degeneracy that arises during our computations is not very severe, we are able to follow Varchenko's 1976 exposition [17]. Modern techniques for the general resolution of singularities may be needed for applications where the domain has a more intricate periodic structure than $\mathbb{Z}^2$, and especially in high-dimensional settings. In those cases one may employ methods and results by Greenblatt [6] in two dimensions or Collins, Greenleaf, and Pramanik [4] in higher dimensions. At one point in Corollary 3.4 we also invoke a recent result by

Ikromov and Müller [7] regarding the stability of degenerate integrals under linear perturbation of the phase.

Unlike in the continuous setting, varying parameters λ_1, λ_2 is not equivalent to performing a diagonal linear transformation on x because the domain $\mathbb{Z}^2$ and its Fourier dual both lack a dilation symmetry. We show in this paper that the dispersion pattern for (1.1) retains the same topological and geometric structure found in [16] for all values of ω, λ_j. In particular there is no choice of parameters that generates exceptional degeneracy of the dispersive estimate or bifurcation of the dynamical system. Separately we show that interactions outside the light cone are all subject to an exponential bound, and that exponential bounds of any desired order are achieved by setting $|y-x|/|t|$ sufficiently large. The latter statement is akin to (and readily implies) Lieb-Robinson bounds (cf. [8], [11], [3], [5], [15], [12], [14]) for the corresponding quantum harmonic lattice. The background and details of this application are presented in Section 5.

Section 2 enumerates the precise statements and illustrations of our main results. The proof is sketched out in the following section, assuming a number of propositions about the critical points of $t\gamma(k) - x\cdot k$. Section 4 takes a detailed look at the Taylor series expansions of γ in order to verify those assertions. The concluding section translates our results about the discrete Klein-Gordon equation into dynamical properties of the quantum harmonic lattice.

2. Main results

In this section we state the result describing the long-time behavior of the solution of (1.1). We start with estimates of a slightly more general oscillatory integral and our main result is based on these estimates.

For the function

$$\gamma(k) = \left(\omega^2 + \sum_{j=1}^{2} 2\lambda_j(1-\cos k_j)\right)^{1/2}, \qquad k=(k_1,k_2)\in[-\pi,\pi]^2, \tag{2.1}$$

introduce

$$I(t,x,\eta) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} e^{i(k\cdot x - t\gamma(k))}\eta(k)\,dk, \tag{2.2}$$

where $x\in\mathbb{Z}^2$, $t\in\mathbb{R}$, and η is a smooth test function on $[-\pi,\pi]^2$ with periodic boundary conditions. The asymptotic behavior of oscillatory integrals is generally influenced by local considerations, in which case $\eta(x)$ may be assumed to have compact support in a fundamental domain of $\mathbb{R}^2/2\pi\mathbb{Z}^2$ (for example as part of a partition of unity). In that case η can be extended by zero to a function on all of $\mathbb{R}^2$ and one may define

$$\tilde{I}(t,x,\eta) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i(k\cdot x - t\gamma(k))}\eta(k)\,dk, \tag{2.3}$$

for all $x\in\mathbb{R}^2$ up to a unimodular constant whose value is exactly 1 if $x\in\mathbb{Z}^2$. It is often convenient to consider x of the form $x=vt$, with v a fixed vector in $\mathbb{R}^2$ (representing velocity), so that the integral (2.3) can be written as

$$\tilde{I}(t,vt,\eta) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{it\phi_v(k)}\eta(k)\,dk, \quad \text{where} \quad \phi_v(k) := k\cdot v - \gamma(k). \tag{2.4}$$

Estimates on $I(t,x,\eta)$ will follow from restricting the corresponding bound on (2.4) to examples with $x = vt\in\mathbb{Z}^2$.

The long-time behavior of (2.4) is dictated by the highest level of degeneracy of the phase within the support of η. In the absence of critical points, integration by parts multiple times yields a rapid (faster than polynomial) decay. In the case where critical points are present but are non-degenerate, the standard stationary phase argument provides $|t|^{-1}$ decay for the integral. Finally, if there are

degenerate critical points, the decay is slower and more careful analysis is needed to determine its exact order. It is easy to see that for any point $k^* \in [-\pi, \pi]^2$, there is a choice of the velocity v such that k^* is a critical point of $\phi_v(k)$ (namely, $v = \nabla\gamma(k^*)$). The order of degeneracy of the phase at that point is determined by second and higher-order derivatives of γ, as the linear component is cancelled by subtracting $k^* \cdot v$. We introduce a partition of $[-\pi, \pi]^2$ with respect to the degeneracy order of γ,

$$[-\pi, \pi]^2 = K_1 \cup K_2 \cup K_3, \tag{2.5}$$

where

$$\begin{aligned} K_1 &= \{k \in [-\pi, \pi]^2 : \det D^2\gamma(k) \neq 0\}, \\ K_2 &= \{k \in [-\pi, \pi]^2 : \det D^2\gamma(k) = 0,\ (\xi \cdot \nabla)^3\gamma(k) \neq 0\}, \\ K_3 &= \{k \in [-\pi, \pi]^2 : \det D^2\gamma(k) = 0,\ (\xi \cdot \nabla)^3\gamma(k) = 0\}. \end{aligned} \tag{2.6}$$

In the definition of K_2 and K_3, ξ stands for an eigenvector of the 2×2 matrix $D^2\gamma(k)$ corresponding to the zero eigenvalue.

The analysis of Section 4 allows us to describe the structure of this partition in detail. Proposition 4.1 notes in particular that the rank of $D^2\gamma(k)$ is never zero, so the direction of ξ is always well defined.

Lemma 2.1. *The sets K_i, $i = 1, 2, 3$, defined in* (2.6) *possess the following properties.*

- *K_3 consists of four points related by mirror symmetry across the coordinate axes.*
- *K_2 consists of two closed curves, one around the origin and the other around the point (π, π), with the four points of K_3 removed from the latter curve.*
- *$K_1 = [-\pi, \pi]^2 \setminus (K_3 \cup K_2)$. This set consists of three open regions: the interior of the small closed curve around zero, the interior of the closed curve around the point (π, π), and the area of the compactified torus enclosed between these two curves.*

The structure of the partition is displayed in Figure 1 (similar to Figure 8).

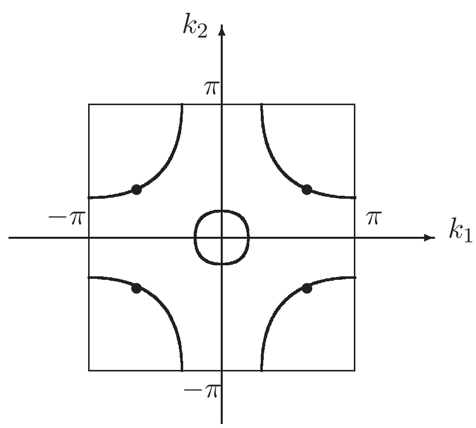

FIGURE 1. Sets K_1, K_2, K_3

Proof. The statements follow immediately from equation (4.5) and Corollary 4.8. □

To see the connection between the velocities and possible degeneracies of ϕ_v, let us analyze images of the sets K_i in the velocity space:

$$\begin{aligned} V_3 &= \{v \in \mathbb{R}^2 : \text{ there exists } k \in K_3 \text{ such that } v = \nabla\gamma(k)\}, \\ V_2 &= \{v \in \mathbb{R}^2 \setminus V_3 : \text{ there exists } k \in K_2 \text{ such that } v = \nabla\gamma(k)\}, \\ V_1 &= \{v \in \mathbb{R}^2 \setminus (V_2 \cup V_3) : \text{ there exists } k \in K_1 \text{ such that } v = \nabla\gamma(k)\}, \\ V_0 &= \{v \in \mathbb{R}^2 : \text{ for all } k \in [-\pi,\pi]^2,\ v \neq \nabla\gamma(k)\}. \end{aligned} \tag{2.7}$$

Alternatively, under the mapping $\mathcal{V} : [-\pi,\pi]^2 \to \mathbb{R}^2$ defined by

$$\mathcal{V}(k) = \nabla\gamma(k), \tag{2.8}$$

sets (2.7) admit the representation

$$\begin{aligned} V_3 &= \mathcal{V}(K_3), \\ V_2 &= \mathcal{V}(K_2), \\ V_1 &= \mathcal{V}(K_1) \setminus (V_3 \cup V_2), \\ V_0 &= \mathbb{R}^2 \setminus \mathcal{V}([-\pi,\pi]^2). \end{aligned} \tag{2.9}$$

The collection $\{V_i\}_{i=0}^3$ forms a partition of $\mathbb{R}^2$. More detailed description of its structure is provided in the following result.

Proposition 2.2. *Let the sets $\{V_i\}_{i=0}^3$ be defined by (2.7). Then they are located as shown on Figure 2. There are two simple closed continuous curves Ψ_1 and Ψ_2 around the origin that split the plane into three open regions. More precisely, Ψ_1 encloses a convex region and Ψ_2 consists of four concave arcs that meet at cusps. The four vertices of these cusps form the set V_3. The union of Ψ_1 and Ψ_2, with the four points that belong to V_3 removed, is V_2. The union of the two inner bounded open regions is V_1. The outer-most unbounded region is V_0 which has boundary Ψ_1.*

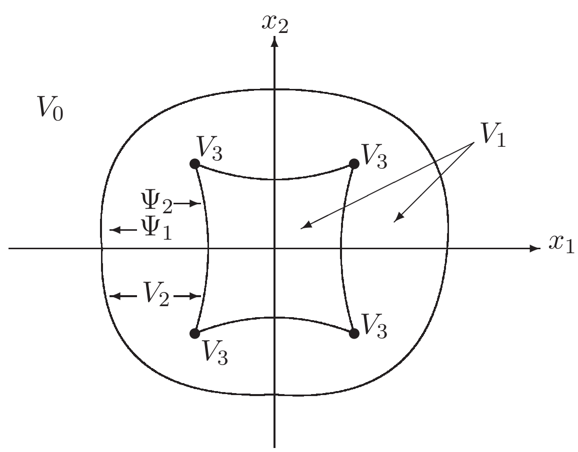

FIGURE 2. Sets V_0, V_1, V_2, and V_3.

We can now state the main result describing the connection between a velocity v and the decay order of an oscillatory integral (2.4) with phase function γ.

Theorem 2.3. *Let the sets $\{V_i\}_{i=1}^3$ be defined by (2.7), η be a smooth periodic function on $[-\pi,\pi]^2$, and integral $I(t,x,\eta)$ defined by (2.2). Then for any fixed $\delta > 0$, there exist constants C_0,C_1, C_2 and C_3 depending on η such that*

$$\text{for all } x \text{ with } \operatorname{dist}(x, tV_3) \le t\delta, \text{ we have } |I(t,x,\eta)| \le \frac{C_3}{|t|^{3/4}}, \tag{2.10}$$

$$\text{for all } x \text{ with } \operatorname{dist}(x, tV_3) > t\delta \text{ and } \operatorname{dist}(x, tV_2) \le t\delta, \text{ we have } |I(t,x,\eta)| \le \frac{C_2}{|t|^{5/6}}, \tag{2.11}$$

$$\text{for all } x \text{ with } \operatorname{dist}(x, t(V_3 \cup V_2)) > t\delta \text{ and } \operatorname{dist}(x, tV_1) \le t\delta, \text{ we have } |I(t,x,\eta)| \le \frac{C_1}{|t|}, \tag{2.12}$$

$$\text{given } N \ge 1, \text{ for all } x \text{ with } \operatorname{dist}(x, t(V_3 \cup V_2 \cup V_1)) > \delta, \text{ we have } |I(t,x,\eta)| \le \frac{C_0}{|t|^N}. \tag{2.13}$$

Each of C_0, C_1, and C_2 depend on δ (and C_0 also depends on N). The values of C_0 – C_2 are expected to approach infinity as δ approaches zero, while C_3 is independent of δ.

If one is only concerned with the worst possible decay among all $x \in \mathbb{Z}^2$, the simpler statement is as follows.

Corollary 2.4. *Let the integral $I(t,x,\eta)$ be defined by (2.2). Then there exists $C = C(\eta) > 0$ such that for all $x \in \mathbb{Z}^2$,*

$$|I(t,x,\eta)| \le \frac{C}{|t|^{3/4}}. \tag{2.14}$$

Corollary 2.5. *As a special case of Theorem 2.3, the propagators of the discrete Klein-Gordon equation (1.1) are recovered by choosing $\eta = \gamma^m$, $m = -1, 0, 1$. Hence the values of $u(x,t)$ and $u_t(x,t)$ for a solution of (1.1) both satisfy (2.10)–(2.13), provided the initial data g, h are supported at the origin. The result extends to all g, h supported in $B(0,R)$ by superposition, once enough time has elapsed that $\delta t > 2R$.*

The values of the exponents in (2.10)–(2.12) are dictated by the worst degeneracy degree of critical points of the phase function. For example, if a velocity belongs to the set V_1 and is relatively far from V_2 and V_3, all the critical points of ϕ_v will be uniformly non-degenerate. In this case the decay rate of an oscillatory integral is $|t|^{-d/2}$ in arbitrary dimension d (hence it is $|t|^{-1}$ in dimension two).

Let us now briefly describe how the rates produced by velocities that are near V_2 and V_3 are computed (see Section 3.1 for details). If $v \in V_2 \cup V_3$, then ϕ_v has at least one degenerate critical point $k^* \in [-\pi,\pi]^2$. Then the Taylor series expansion of ϕ_v near its critical point takes the form

$$\phi_v(k) = k \cdot v - \gamma(k) = c_0 + \sum_{\substack{n,m \ge 0 \\ n+m \ge 2}} c_{n,m}(k_1 - k_1^*)^n (k_2 - k_2^*)^m. \tag{2.15}$$

This Taylor series is said to be supported on the set of indices (m,n) where $c_{m,n} \ne 0$. Roughly speaking, one computes the leading-order decay of the corresponding oscillatory integral by measuring the distance from the origin to the convex hull of the Taylor series support, then taking the reciprocal. However the support is not invariant under changes of coordinates, so one must first choose an "adapted" coordinate system that maximizes this distance [17]. It turns out that for each function $\phi_v(k)$ the linear coordinate system that diagonalizes the Hessian matrix is adapted (Lemma 4.9 verifies this property in the one case where it is not readily apparent). According to the definitions (2.6) and (2.7), the Taylor series of ϕ_v with respect to these coordinates has more vanishing low-order terms if $v \in V_3$ as compared to $v \in V_2$. Therefore its Newton polyhedron lies further away from the origin, and the oscillatory integral decays more slowly. The Newton

polyhedra associated with $v \in V_2$ and $v \in V_3$ are sketched in Figures 3 and 4 and give rise to the exponents in (2.11) and (2.10) respectively.

If $0 < \operatorname{dist}(v, V_2 \cup V_3) < \delta$, then ϕ_v does not have a degenerate critical point itself, but it is related to the degenerate phase functions described above by a small linear perturbation. The fact that oscillatory integral estimates are stable under such perturbations is proved in [7].

According to its definition, $V_0 \subset \mathbb{R}^2$ consists of velocities that produce phase functions with no critical points in the domain of integration. As a result one can recover polynomial decay (in x) of $I(t, x, \eta)$ of any order by repeated integrations by parts. In fact for solutions of the discrete Klein-Gordon equation the decay satisfies a number of exponential bounds.

Theorem 2.6. *For every $\mu > 0$ there exists constants $0 < v_\mu \le \frac{1}{\mu}(1 + 2\sqrt{\lambda_1 + \lambda_2}\,\sinh(\mu/2))$ and $C_\mu < \omega + 2\sqrt{\lambda_1 + \lambda_2}\cosh(\mu/2)$ such that*

$$\Big|\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \cos(t\,\gamma(k))e^{ik\cdot x}\,dk\Big| \le e^{-\mu(|x| - v_\mu|t|)} \tag{2.16}$$

$$\Big|\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{\sin(t\,\gamma(k))}{\gamma(k)}e^{ik\cdot x}\,dk\Big| \le e^{-\mu(|x| - v_\mu|t|)} \tag{2.17}$$

$$\Big|\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \gamma(k)\sin(t\,\gamma(k))e^{ik\cdot x}\,dk\Big| \le C_\mu e^{-\mu(|x| - v_\mu|t|)} \tag{2.18}$$

The upper bound for v_μ as stated in Theorem 2.6 behaves as expected for large μ (see Corollary 2.2 in [12]) but it has some evident drawbacks over the rest of the range. First, the sharp value of v_μ must be an increasing function of μ so the apparent asymptote as $\mu \to 0$ is an artifact of the calculation. In addition the estimates (2.16) and (2.17) don't show any time-decay when applied to a point $x \in tV_0$ with $\frac{|x|}{t} \le v_\mu$. The last result shows that in fact every $x \in tV_0$ is subject to an effective exponential bound.

Theorem 2.7. *Let V_0 be the set defined in* (2.7)*. For any $x \in \mathbb{R}^2$ with $\frac{x}{t} \in V_0$ there exists $\mu > 0$ and constants $C_1 < \infty$, $C_2 \le \sqrt{\omega^2 + 4(\lambda_1 + \lambda_2)}$ such that*

$$\Big|\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \cos(t\,\gamma(k))e^{ik\cdot x}\,dk\Big| \le e^{-\mu\,\operatorname{dist}(x,\,tV_1)} \tag{2.19}$$

$$\Big|\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{\sin(t\,\gamma(k))}{\gamma(k)}e^{ik\cdot x}\,dk\Big| \le C_1 e^{-\mu\,\operatorname{dist}(x,\,tV_1)} \tag{2.20}$$

$$\Big|\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \gamma(k)\sin(t\,\gamma(k))e^{ik\cdot x}\,dk\Big| \le C_2 e^{-\mu\,\operatorname{dist}(x,\,tV_1)} \tag{2.21}$$

3. Proof of the main results

3.1. **Proof of Theorem 2.3.** The material of this section is presented in the following order: we start with some background information on oscillatory integrals, followed by the main local results (Lemma 3.1 and Corollary 3.4), and the proof of Theorem 2.3 is then obtained from local estimates through a partition of unity argument.

Consider an oscillatory integral in several variables

$$I(t, \eta) = \int_{\mathbb{R}^d} e^{it\phi(k)}\eta(k)dk, \tag{3.1}$$

where η is supported in a neighborhood of an isolated critical point k^* of ϕ (we follow the notation introduced in [17] and [6]). When it is convenient to do so, one may apply an affine translation so that the critical point is located at the origin. If $\operatorname{supp}(\eta)$ is small enough, $I(t, \eta)$ has an asymptotic

expansion

$$I(t,\eta) \approx e^{it\phi(0)} \sum_{j=0}^{\infty} (d_j(\eta) + d'_j(\eta)\ln(t)) t^{-s_j}, \quad \text{as } t \to \infty, \tag{3.2}$$

where s_j is an increasing arithmetic progression of positive rational numbers independent of η. The oscillatory index of the function ϕ at k^* is defined to be the leading-order exponent s_0. We assume that s_0 is chosen to be minimal such that in any sufficiently small neighborhood U containing k^* either $d_0(\eta)$ or $d'_0(\eta)$ is nonzero for some η supported in U.

Estimates (2.10) and (2.11) are essentially statements about the oscillatory index of $\phi_v(k)$ at its critical points for different values of v. The following algorithm assists in their computation.

Suppose ϕ is analytic with a critical point at k^*. Locally there is a Taylor series expansion $\phi(k) = c_0 + \sum c_n (k - k^*)^n$, with the sum ranging over all $n \in \mathbb{Z}_+^d$ with $n_1 + \ldots + n_d \geq 2$. Let $K \subset \mathbb{Z}^+$ be the collection of all indices n for which $c_n \neq 0$.

Newton's polyhedron associated to ϕ at its critical point k^* is defined as the convex hull of the set

$$\bigcup_{n \in K} (n + \mathbb{R}_+^d),$$

where $\mathbb{R}_+^d$ is the positive octant $\{x \in \mathbb{R}^d : x_j \geq 0 \text{ for } 1 \leq j \leq d\}$. We denote Newton's polyhedron of ϕ by $N_+(\phi)$. Newton's diagram of ϕ is the union of all compact faces of $N_+(\phi)$. Finally, the Newton distance $d(\phi)$ is defined as $d(\phi) = \inf\{t : (t,t) \in N_+(\phi)\}$.

Note that the vanishing of Taylor coefficients is affected by changes to the underlying coordinates, thus each local coordinate system y generates its own sets K^y and $N_+^y(\phi)$ and Newton distance $d^y(\phi)$. Define the height of an analytic function ϕ at its critical point to be $h(\phi) := \sup\{d^y(\phi)\}$, with the supremum taken over all local coordinate systems y. A coordinate system y is called adapted if $d^y(\phi) = h(\phi)$.

It was shown in [17] (p. 177, Theorem 0.6) that under some natural assumptions, the oscillation index s_0 of a function ϕ is equal to $1/h(\phi)$.

We now compute the height of the phase of the integral $I(t,x,\eta)$ at its critical point(s). Recall that with the notation $v = \frac{x}{t}$, the phase of $I(t,x,\eta)$ can be written in the form (2.15). Given a point $k^* \in [-\pi,\pi]^2$, the value $v = \nabla\gamma(k^*) \in \mathbb{R}^2$ is the unique choice for which ϕ_v has a critical point at k^*. Denote the height of this ϕ_v at k^* by $h(k^*)$.

Lemma 3.1. *The height function $h(k^*)$, defined above, is constant on each of the sets $\{K_i\}_{i=1}^3$ defined in* (2.6) *with the following values:*

(1) *if $k^* \in K_1$, then $h(k^*) = 1$,*
(2) *if $k^* \in K_2$, then $h(k^*) = 6/5$,*
(3) *if $k^* \in K_3$, then $h(k^*) = 4/3$.*

Proof. Note that $\gamma(k)$ and $\phi_v(k)$ differ by a linear function, so their derivatives coincide except at the first order. In the simpler case $k^* \in K_1$, $\det D^2\gamma(k^*) = \det D^2\phi_v(k^*) \neq 0$ by definition. Moreover, the determinant of the Hessian of ϕ_v at k^* remains non-zero in any local coordinate system, thus $h(k^*) = d/2 = 1$.

Next consider $k^* \in K_2$. In this case $\det D^2\gamma(k^*) = 0$ and $(\xi \cdot \nabla)^3\gamma(k^*) \neq 0$, where ξ is an eigenvector of $D^2\gamma(k^*)$ corresponding to the zero eigenvalue. The mixed second-order derivative vanishes because $D^2\gamma(k^*)$ has orthogonal eigenvectors, and by Proposition 4.1 it is guaranteed that $(\xi^\perp \cdot \nabla)^2\gamma(k^*) \neq 0$. This information suffices to compute the Newton's distance of ϕ_v at k^* in the linear coordinate system with axes $\{\xi^\perp, \xi\}$ and given by coordinates $k - k^* = y_1\xi^\perp + y_2\xi$.

The associated Newton's polyhedron is of the form displayed on Figure 3, with the Newton's distance being 6/5.

In order to show that $\{y_1, y_2\}$ is an adapted coordinate system, and therefore $h(k^*) = 6/5$, we use the following result from Varchenko:

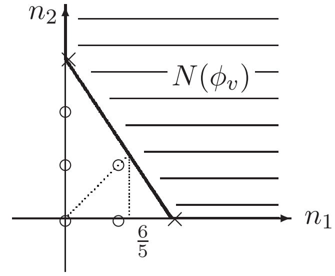

FIGURE 3. Newton's polyhedron and Newton's distance of the Taylor series corresponding to ϕ_v for $k^* \in K_2$

Proposition 3.2 ([17, part 2 of Proposition 0.7])**.** *Assume that for a given series $f = \sum c_n y^n$, the point $(d(f), d(f))$ lies on a closed compact face Γ of the Newton's polyhedron. Let $a_1 n_1 + a_2 n_2 = m$ be the equation of the straight line on which Γ lies, where a_1, a_2, and m are integers and a_1 and a_2 are relatively prime. Then the coordinate system y is adapted if both numbers a_1 and a_2 are larger than 1.*

The Newton polyhedron displayed on Figure 3 has only one compact face (that also contains the point $(d(\phi_v), d(\phi_v)))$, which lies on the line with the equation $3n_1 + 2n_2 = 6$. Since 2 and 3 are relatively prime, the coordinate system is adapted by Proposition 3.2.

Finally, let $k \in K_3$. To provide a more concise notation, let ∂_ξ and $\partial_{\xi^\perp}$ indicate the directional derivatives $\xi \cdot \nabla$ and $\xi^\perp \cdot \nabla$ respectively. These also serve as partial derivatives ∂_{y_2} and ∂_{y_1} with respect to coordinates $\{y_1, y_2\}$. By definition of K_3 we have

$$\partial_\xi^2 \gamma(k^*) = 0, \qquad \partial_\xi \partial_{\xi^\perp} \gamma(k^*) = 0, \qquad \partial_{\xi^\perp}^2 \gamma(k^*) \neq 0, \qquad \text{and} \qquad \partial_\xi^3 \gamma(k^*) = 0. \tag{3.3}$$

On the other hand, one can show that $\partial_\xi^4 \gamma(k^*)$ and $\partial_\xi^2 \partial_{\xi^\perp} \gamma(k^*)$ do not both vanish (see Lemma 4.9). The possible Newton's polyhedra that arise are indicated in Figure 4, however, the Newton distance is equal to 4/3 in all situations:

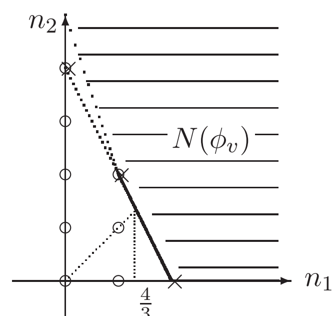

FIGURE 4. Possible Newton's polyhedra and Newton's distance of the Taylor series corresponding to ϕ_v for $k^* \in K_3$

Moreover, in all situations the face containing the point $(d(\phi_v), d(\phi_v))$ lies on the line with the equation $2n_1 + n_2 = 4$, and Proposition 3.2 does not apply. To verify that the system $\{y_1, y_2\}$ is adapted in this case, we use a different result from Varchenko:

Proposition 3.3 ([17, Proposition 0.8])**.** *Assume that for a given series $f = \sum c_n y^n$, the point $(d(f), d(f))$ lies on a closed compact face Γ of the Newton's polyhedron. Let $a_1 n_1 + n_2 = m$ be the equation of the straight line on which Γ lies, where a_1 and m are integers. Let*

$$f_\Gamma(y) = \sum_{n \in \Gamma} c_n y^n \quad \textit{and} \quad P(y_1) = f_\Gamma(y_1, 1). \tag{3.4}$$

If the polynomial P does not have a real root of multiplicity larger than $m(1+a_2)^{-1}$, then y is a coordinate system adapted to f.

For the face Γ, displayed on Figure 4, we have

$$
\begin{aligned}
f_\Gamma(y) &= \frac{\partial^2_{\xi^\perp}\gamma(k^*)}{2}y_1^2 + \frac{\partial^2_\xi\partial_{\xi^\perp}\gamma(k^*)}{2}\,y_1y_2^2 + \frac{\partial^4_\xi\gamma(k^*)}{24}\,y_2^4,\\
P(y_1) &= \frac{\partial^2_{\xi^\perp}\gamma(k^*)}{2}y_1^2 + \frac{\partial^2_\xi\partial_{\xi^\perp}\gamma(k^*)}{2}\,y_1 + \frac{\partial^4_\xi\gamma(k^*)}{24}.
\end{aligned}
\tag{3.5}
$$

The discriminant of P is

$$
\mathcal{D} = \left(\frac{\partial^2_\xi\partial_{\xi^\perp}\gamma(k^*)}{2}\right)^2 - 4\frac{\partial^4_\xi\gamma(k^*)}{24}\frac{\partial^2_{\xi^\perp}\gamma(k^*)}{2} = \frac{1}{12}\left(3(\partial^2_\xi\partial_{\xi^\perp}\gamma(k^*))^2 - \partial^2_{\xi^\perp}\gamma(k^*)\partial^4_\xi\gamma(k^*)\right). \tag{3.6}
$$

It follows from Lemma 4.9 that this discriminant is nonzero whenever $k^*\in K_3$, thus P can have real roots of multiplicity at most one. On the other hand, $m(1+a_2)^{-1} = 4/3$, and by Proposition 3.3 the coordinate system $\{y_1, y_2\}$ is adapted. □

As was stated earlier, the height of the phase function determines the decay order of an oscillatory integral in a neighborhood of its critical point. In [17] it is shown that the oscillation index of a phase ϕ is equal to $1/h(\phi)$, giving both upper and lower bounds for the decay rate. More recently, Ikromov and Müller in [7] showed that the upper bound is stable under linear perturbations of the phase function. Their result, combined with Lemma 3.1, brings us the following

Corollary 3.4. *Let sets $\{K_i\}_{i=1}^3$ be defined by (2.6) and fix $k^*\in[-\pi,\pi]^2$. Then there exist a neighborhood of k^*, Ω_{k^*}, and a positive constant C_{k^*} such that for all η supported in Ω_{k^*},*

$$
\left|\tilde I(t,x,\eta)\right| \le C_{k^*}\,\|\eta\|_{C^3(\mathbb{R}^2)}\,\frac{1}{|t|^{3/4}}, \qquad \text{if } k^*\in K_3, \tag{3.7}
$$

$$
\left|\tilde I(t,x,\eta)\right| \le C_{k^*}\,\|\eta\|_{C^3(\mathbb{R}^2)}\,\frac{1}{|t|^{5/6}}, \qquad \text{if } k^*\in K_2, \tag{3.8}
$$

$$
\left|\tilde I(t,x,\eta)\right| \le C_{k^*}\,\|\eta\|_{C^3(\mathbb{R}^2)}\,\frac{1}{|t|}, \qquad \text{if } k^*\in K_1. \tag{3.9}
$$

for all $x\in\mathbb{R}^2$.

Proof. By a direct consequence of Theorem 1.1 in [7], for a point $k^*\in[-\pi,\pi]^2$, there exist a neighborhood Ω_{k^*} and a positive constant C_{k^*} such that

$$
\left|\int_{\Omega_{k^*}} e^{it(\phi_v(k)+x\cdot k)}\eta(k)\,dk\right| \le C_{k^*}\,\|\eta\|\,\frac{1}{|t|^{h(k^*)}}, \tag{3.10}
$$

for all $x\in\mathbb{R}^2$ and η supported in Ω_{k^*}. This, together with the result of Lemma 3.1, proves the claim. □

We can extract some additional important information about the neighborhoods Ω_{k^*}, introduced in Corollary 3.4, that will be useful for the main part of the proof. Specifically, if $k^*\in K_1$ then the corresponding neighborhood Ω_{k^*} does not contain any points from $K_2\cup K_3$, and if $k^*\in K_2$ then Ω_{k^*} is disjoint from K_3. To prove the latter claim, suppose there is a point $k_0\in K_3$ that also belongs to Ω_{k^*}. Then with $v=\nabla\gamma(k_0)$ the oscillation index of ϕ_v in a neighborhood of k_0 is equal to 3/4. For η supported in a small neighborhood of k_0 inside of Ω_{k^*}, and $x=tv$, the asymptotic lower bound dictated by (3.2) and statement (3) of Lemma 3.1 contradicts the decay rates of (3.8) and (3.9).

At last, we need the following well-known estimate.

Lemma 3.5. *If* $\operatorname{supp}\eta$ *does not contain any critical points of the phase function* $k\cdot x - t\gamma(k)$ *then for any* $M>0$,

$$|\tilde{I}(t,x,\eta)| \le C(M,\eta,d)\frac{1}{|t|^M}. \tag{3.11}$$

where d *is the infimum of* $|x/t-\nabla\gamma(k)|$ *over the support of* η.

Proof of Theorem 2.3. Fix $\delta>0$. The nonstationary phase bound (2.13) follows immediately from the construction, as the gradient of $x\cdot k - t\gamma(k)$ must have magnitude at least $\operatorname{dist}(x,tV_1)$.

To prove (2.10), Take the system of neighborhoods $\{\Omega_k\}_{k\in[-\pi,\pi]^2}$ described in Corollary 3.4. By construction $\{\Omega_k\}_{k\in[-\pi,\pi]^2}$ covers $[-\pi,\pi]^2$ and we can choose a finite sub-cover, say $\{\Omega_j\}_{j=1}^{N_0}$. Now, let a collection of smooth functions $\{\omega_j\}_{j=1}^{N_0}$ form a partition of unity with respect to $\{\Omega_j\}_{j=1}^{N_0}$, then

$$|I(t,x,\eta)| \le \sum_j |I(t,x,\eta_j)| = \sum_j |\tilde{I}(t,x,\eta_j)|, \tag{3.12}$$

where $\eta_j = \eta\,\omega_j$, is supported in Ω_j. Since for t away from zero every integral that satisfies (3.8) or (3.9) also satisfies (3.7), and all three are uniformly bounded for all times, we have

$$|I(t,x,\eta)| \le \sum_j C_j\,\|\eta_j\|_{C^3(\mathbb{R}^2)}\,\frac{1}{|t|^{3/4}} = C(\eta)\frac{1}{|t|^{3/4}}. \tag{3.13}$$

Note that even though (3.13) holds for all $x\in\mathbb{R}^2$, better estimates are available when x is removed from tV_3.

To prove the estimates (2.11) and (2.12) we need to refine our construction of the cover so that the $|t|^{-3/4}$ bound in (3.7) is never invoked (in the latter case one should also avoid applying (3.8)). The following construction will suit both situations.

The function $\mathcal{V}$, defined by (2.8), is uniformly continuous on $[-\pi,\pi]^2$, so we can choose an $0<\epsilon=\epsilon(\delta)<\pi/2$ such that

$$\operatorname{diam}(\mathcal{V}(B_\epsilon)) < \delta/2$$

for every ball B_ϵ of radius ϵ. At each $k\in[-\pi,\pi]^2$ define a smaller δ-dependent neighborhood

$$\Omega_k(\delta) = \Omega_k \cap B_\epsilon(k), \tag{3.14}$$

where Ω_k is again as in Corollary 3.4. As before, pick a finite sub-collection of $\{\Omega_k(\delta)\}_{k\in[-\pi,\pi]^2}$ that is also a cover of $[-\pi,\pi]^2$, say $\{\Omega_{k_j}(\delta)\}_{j=1}^N$, and generate a partition of unity ω_j subordinate to this cover. For simplicity of notation we will write $\Omega_j = \Omega_{k_j}(\delta)$ with $j\in\{1,2,\ldots,N\}$.

Sort the neighborhoods Ω_j according to the location of their "center" point k_j. For each $m = 1,2,3$ let $J_m := \{j\in\{1,\ldots,N\} : k_j\in K_m\}$, where K_m are the sets defined in (2.6). The discussion following Corollary 3.4 indicates that

$$\Big(\bigcup_{j\in J_1\cup J_2}\Omega_j\Big)\cap K_3 = \emptyset \quad \text{and} \quad \Big(\bigcup_{j\in J_1}\Omega_j\Big)\cap K_2 = \emptyset. \tag{3.15}$$

Suppose $x\in\mathbb{Z}^2$ is chosen so that $\operatorname{dist}(x,tV_3) > t\delta$. In other words, $|x/t-\nabla\gamma(k^*)|>\delta$ for any $k^*\in K_3$. Moreover $|x/t-\nabla\gamma(k)|>\delta/2$ for all $k\in\bigcup_{j\in J_3}\Omega_j$ because each neighborhood has radius at most ϵ. Split the sum (3.12) into two parts

$$|I(t,x,\eta)| \le \sum_{j\in J_3}|\tilde{I}(t,x,\eta_j)| + \sum_{j\notin J_3}|\tilde{I}(t,x,\eta_j)|. \tag{3.16}$$

Lemma 3.5 applies to each term in the first sum, with $d=\delta/2$. Terms in the second sum are bounded by (3.8) or (3.9). The slowest time-decay out of these has the rate $|t|^{-5/6}$ from (3.8), which verifies (2.11).

The argument is similar if $\operatorname{dist}(x, t(V_3 \cup V_2)) > t\delta$. One splits (3.12) in the parts

$$|I(t,x,\eta)| \le \sum_{j\in J_2\cup J_3} |\tilde{I}(t,x,\eta_j)| + \sum_{j\notin J_2\cup J_3} |\tilde{I}(t,x,\eta_j)|, \tag{3.17}$$

and once again Lemma 3.5 applies to each term in the first sum, with $d = \delta/2$, and terms in the second sum are bounded by (3.9). This is sufficient to verify (2.12), completing the proof of Theorem 2.3. $\square$

3.2. **Proof of Exponential Bounds.**

Proof of Theorem 2.6. Note that $\gamma^2(k)$ extends to a complex-analytic function on $k \in \mathbb{C}^2$ that is periodic under the shifts $k_j \to k_j + 2\pi$, $j = 1, 2$. After composition with the holomorphic map $\cos(t\sqrt{z})$, the same is true of $\cos(t\,\gamma(k))$. By shifting the contour of integration for k_1 and k_2, the left-hand quantity in (2.16) is equal to

$$\begin{aligned}\frac{e^{-\mu|x|}}{(2\pi)^2}\Big|\int_{[-\pi,\pi]^2} \cos(t\,\gamma(k+i\mu\tfrac{x}{|x|}))e^{ik\cdot x}\,dk\Big| &\le \max_{k\in[-\pi,\pi]^2} \big|\cos(t\,\gamma(k+i\mu\tfrac{x}{|x|}))\big|e^{-\mu|x|}\\ &\le \max_{k\in[-\pi,\pi]^2} e^{|\operatorname{Im} t\,\gamma(k+i\mu\frac{x}{|x|})|}e^{-\mu|x|}\\ &= e^{-\mu(|x|-v_\mu|t|)}\end{aligned}$$

where $v_\mu = \mu^{-1}\max\{|\operatorname{Im}\gamma(k+i\tilde{k})| : k \in [-\pi,\pi]^2, |\tilde{k}| = \mu\}$. Referring back to the definition of $\gamma(k)$ in (2.1), one obtains a bound $v_\mu \le \frac{2}{\mu}\sqrt{\lambda_1+\lambda_2}\,\sinh(\mu/2)$ by applying the inequality $|\operatorname{Im}\sqrt{z^2+w^2}| \le \sqrt{(\operatorname{Im} z)^2 + (\operatorname{Im} w)^2}$ for pairs of complex numbers.

The same argument applies to the sine propagator as well, thanks to the bound $\big|\frac{\sin z}{z}\big| \le e^{|\operatorname{Im} z|}$. By shifting the integration contour as above, the left-hand quantity in (2.17) is equal to

$$\begin{aligned}t\frac{e^{-\mu|x|}}{(2\pi)^2}\Big|\int_{[-\pi,\pi]^2} \frac{\sin(t\,\gamma(k+i\mu\frac{x}{|x|}))}{t\,\gamma(k+i\mu\frac{x}{|x|})}e^{ik\cdot x}\,dk\Big| &\le t\max_{k\in[-\pi,\pi]^2} e^{|\operatorname{Im} t\,\gamma(k+i\mu\frac{x}{|x|})|}e^{-\mu|x|}\\ &\le e^{-\mu(|x|-v_\mu|t|)}\end{aligned}$$

for any $v_\mu > \mu^{-1}(1+\max\{|\operatorname{Im}\gamma(k+i\tilde{k})| : k \in [-\pi,\pi]^2, |\tilde{k}| = \mu\})$.

The computation for (2.18) is essentially identical to (2.16) except that it contains an extra factor of $\max\{|\gamma(k+i\tilde{k}) : |\tilde{k}| = \mu\}$, estimated here by $\omega + 2\sqrt{\lambda_1+\lambda_2}\,\cosh(\mu/2)$. $\square$

Proof of Theorem 2.7. It will be important to note that V_0 is the complement of a convex subset of the plane. This is stated as part of Proposition 2.2 and will be proved in Section 4.

For each $\tilde{\mu} \in \mathbb{R}^2$, shifting contours of integration into the complex plane leads to the bound

$$\begin{aligned}\Big|\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \cos(t\,\gamma(k))e^{ik\cdot x}\,dk\Big| &= \frac{e^{-\tilde{\mu}\cdot x}}{(2\pi)^2}\Big|\int_{[-\pi,\pi]^2} \cos(t\,\gamma(k+i\tilde{\mu}))e^{ik\cdot x}\,dk\Big|\\ &\le \max_{k\in[-\pi,\pi]^2} e^{|\operatorname{Im} t\,\gamma(k+i\tilde{\mu})|}e^{-\tilde{\mu}\cdot x}\end{aligned}$$

By assumption $\frac{x}{t}$ lies outside the convex balanced compact set $\{\nabla\gamma(k) : k \in [-\pi,\pi]^2\} = \overline{V}_1$. The complex derivative of γ indicates that $\operatorname{Im}\gamma(k+i\tilde{\mu}) = (\nabla\gamma(k))\cdot\tilde{\mu} + o(|\tilde{\mu}|)$, and the implicit constant in $o(|\tilde{\mu}|)$ converges uniformly across $k \in [-\pi,\pi]^2$. Choose $\mu > 0$ small enough so that

$$\big|\operatorname{Im}\gamma(k+i\tilde{\mu}) - \tilde{\mu}\cdot(\nabla\gamma(k))\big| \le \frac{1}{2}\operatorname{dist}(\tfrac{x}{t}, V_1)|\tilde{\mu}|$$

whenever $|\tilde{\mu}| = 2\mu$. Then

$$\begin{aligned} \left| \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \cos(t\,\gamma(k)) e^{ik\cdot x}\, dk \right| &\le \inf_{|\tilde{\mu}|=2\mu} \max_{k\in[-\pi,\pi]^2} e^{|(t\nabla\gamma(k))\cdot\tilde{\mu}|} e^{\operatorname{dist}(x,\,tV_1)\mu} e^{-x\cdot\tilde{\mu}} \\ &= \inf_{|\tilde{\mu}|=2\mu} \max_{k\in[-\pi,\pi]^2} e^{(t\nabla\gamma(k)-x)\cdot\tilde{\mu}} e^{\operatorname{dist}(x,\,tV_1)\mu} \\ &= e^{-2\mu\operatorname{dist}(x,\,tV_1)} e^{\operatorname{dist}(x,\,tV_1)\mu}. \end{aligned}$$

The first equality follows from the fact that $\nabla\gamma(k)$ is an odd function, so the absolute value can be optimized with either sign. The second equality asserts a geometric principle that given a closed convex set S and a point $x \notin S$,

$$\sup_{y\in S} (y-x)\cdot v \ge -|v| \operatorname{dist}(x,S)$$

with equality taking place if $y \in S$ minimizes the distance and v is parallel to $x-y$. The argument for the sine propagator is essentially identical as the extra factor of t that also appears in the proof of Theorem 2.6 can be overcome by choosing a slightly smaller value of $\mu > 0$ and introducing a large constant C_1. The value of C_2 is limited by estimating the maximum of $|\gamma(k+i\tilde{k})|$ over $|\tilde{k}| \ll 1$. □

4. Properties of the phase function

Throughout this section we will be using the variables (a,b) defined by

$$a(k) = \cos k_1, \quad b(k) = \cos k_2, \tag{4.1}$$

along with $k = (k_1,k_2)$. The function γ, defined in (2.1), as a function of (a,b) takes the form

$$\gamma(a,b) = \left(\omega^2 + 2\lambda_1(1-a) + 2\lambda_2(1-b)\right)^{1/2}. \tag{4.2}$$

A straightforward calculation shows that the Hessian matrix of γ can be written in the form

$$D^2\gamma(k) = \frac{1}{\gamma^3(k)} \begin{pmatrix} \lambda_1 a\gamma^2(k) - \lambda_1^2(1-a^2) & -\lambda_1\lambda_2 \sin k_1 \sin k_2 \\ -\lambda_1\lambda_2 \sin k_1 \sin k_2 & \lambda_2 b\gamma^2(k) - \lambda_2^2(1-b^2) \end{pmatrix} \tag{4.3}$$

and, thus,

$$\begin{aligned} \det D^2\gamma(k) &= \frac{\lambda_1\lambda_2}{\gamma^4(k)} \left(ab\gamma^2(k) - \lambda_1 b(1-a^2) - \lambda_2 a(1-b^2)\right) \\ &= \frac{\lambda_1\lambda_2}{\gamma^4(k)} \left(ab\omega^2 - \lambda_1 b(1-a)^2 - \lambda_2 a(1-b)^2\right). \end{aligned} \tag{4.4}$$

Proposition 4.1. *For every choice of ω, $\lambda_1, \lambda_2 > 0$, there is no $k \in [-\pi,\pi]^2$ such that $D^2\gamma(k)$ is the zero matrix.*

Proof. The off-diagonal entries vanish only if $\sin k_1 = 0$ or $\sin k_2 = 0$. Without loss of generality, suppose $\sin k_1 = 0$. Then $a = \cos k_1 = \pm 1$, so that $\partial_{k_1}^2 \gamma(k) = \pm\lambda_1\gamma^{-1}(k) \neq 0$. If $\sin k_2 = 0$, then $\partial_{k_2}^2\gamma(k)$ is nonzero for similar reasons. □

One of our goals is to describe the set of zeros of the Hessian determinant,

$$\Phi_1 = \{k \in [-\pi,\pi]^2 : \det D^2\gamma(k) = 0\}. \tag{4.5}$$

However, it will be convenient to first study the zeros of $\det D^2\gamma$ as a function of (a,b):

$$\Gamma_1 = \{(a,b) \in [-1,1]^2 : \det D^2\gamma(a,b) = 0\}. \tag{4.6}$$

Using the notation

$$F(a,b) = ab\gamma^2(a,b) - \lambda_1 b(1-a^2) - \lambda_2 a(1-b^2)$$
$$= ab\omega^2 - \lambda_1 b(1-a)^2 - \lambda_2 a(1-b)^2, \tag{4.7}$$

we have that $\det D^2\gamma(a,b) = 0$ if and only if $F(a,b) = 0$ (since $\lambda_1, \lambda_2, \gamma(k) \neq 0$), and therefore,

$$\Gamma_1 = \{(a,b) \in [-1,1]^2 : F(a,b) = 0\}. \tag{4.8}$$

Sometimes it will be convenient to treat F as a function of k, and in those cases we will keep the same notation, $F = F(k)$. Note also that with the notation (4.7) the Hessian matrix takes the form:

$$D^2\gamma(k) = \frac{1}{\gamma^3(k)} \begin{pmatrix} \lambda_1 \partial_b F & -\lambda_1\lambda_2 \sin k_1 \sin k_2 \\ -\lambda_1\lambda_2 \sin k_1 \sin k_2 & \lambda_2 \partial_a F \end{pmatrix}. \tag{4.9}$$

Lemma 4.2. *The equation $F(a,b) = 0$ defines an implicit function $b = B_F(a)$ in $[-1,1]^2$ that is locally continuously differentiable at any $(a,b) \in \Gamma_1$, $|b| \neq 1$. It has the following properties:*

(1) *For any $a \in [-1,1]$, there exists at most one $b \in [-1,1]$ so that $F(a,b) = 0$.*
(2) *For any $(a,b) \in \Gamma_1 \backslash \{(0,0)\}$, $|b| \neq 1$,*

$$\frac{dB_F}{da}(a,b) = -\frac{\lambda_1}{\lambda_2}\frac{b^2(1-a^2)}{a^2(1-b^2)} \leq 0, \tag{4.10}$$

and

$$\frac{dB_F}{da}(0,0) = -\frac{\lambda_2}{\lambda_1}. \tag{4.11}$$

(3) *The set Γ_1 defined in (4.8) is the graph of $b = B_F(a)$, which consists of two continuous arcs $\Gamma_1^1 \cup \Gamma_1^2$ as displayed in Figure 5. The first arc, Γ_1^1, is located in the first quadrant and is convex. The second arc, Γ_1^2, passes through the second and fourth quadrant and is concave.*

The equation $F(a,b) = 0$ also defines an implicit function $a = A_F(b)$. All of the statements in this lemma remain true if (a, A_F, λ_1) and (b, B_F, λ_2) switch roles.

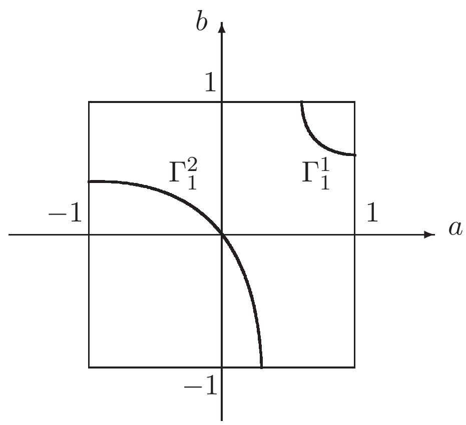

Figure 5. Set Γ_1

Proof. For fixed $a \in [-1,1]$ the function $F(a,b)$ is quadratic with respect to b and has at most one solution in the interval $b \in [-1,1]$. The same is true if b is held fixed and one seeks the value $a = A_F(b)$ for which $F(A_F(b), b) = 0$. As a result B_F is a well-defined function over some subset of $[-1,1]$ and A_F serves as its inverse.

One can write out the value of $B_F(a)$ explicitly using the quadratic formula and derive all the stated properties from this expression. We present a more general approach here in preparation for subsequent computations where an exact formula is not readily available.

The function F is continuously differentiable and

$$\begin{aligned}\frac{\partial F}{\partial b}(a,b) = a\omega^2 - \lambda_1(1-a)^2 + 2\lambda_2 a(1-b) &= \frac{1}{b}\left(ab\omega^2 - \lambda_1 b(1-a)^2\right) + 2\lambda_2 a(1-b) \\ &= \frac{1}{b}\left(F(a,b) + \lambda_2 a(1-b)^2\right) + 2\lambda_2 a(1-b).\end{aligned}$$

Therefore,

$$b\frac{\partial F}{\partial b}(a,b) = \lambda_2 a(1-b^2) \quad \text{for all } (a,b) \in \Gamma_1, \tag{4.12}$$

and this quantity is nonzero so long as $a \neq 0$ and $|b| < 1$. When $a = 0$ one can compute directly that $\frac{\partial F}{\partial b}(0,b) = -\lambda_1 \neq 0$. Therefore

$$\left.\frac{\partial F}{\partial b}(a,b)\right|_{\Gamma_1} = 0 \quad \text{if and only if } |b| = 1.$$

An identical argument applied to the variable a shows that

$$a\frac{\partial F}{\partial a}(a,b) = \lambda_1 b(1-a^2) \quad \text{for all } (a,b) \in \Gamma_1 \tag{4.13}$$

and furthermore that $\left.\frac{\partial F}{\partial a}(a,b)\right|_{\Gamma_1} = 0$ if and only if $|a| = 1$.

In order to prove equation (4.10), we differentiate $F(a,b) = 0$ implicitly with respect to a. Taking advantage of (4.12) and (4.13) the resulting expression reduces to

$$\frac{db}{da} = -\frac{b}{a}\left(\frac{a\,\partial_a F(a,b)}{b\,\partial_b F(a,b)}\right) = -\frac{\lambda_1 b^2(1-a^2)}{\lambda_2 a^2(1-b^2)} \tag{4.14}$$

for all $(a,b) \in \Gamma_1$ away from the origin. At the origin, (4.11) is obtained directly from the facts that $\frac{\partial F}{\partial a}(a,0) = -\lambda_2$ and $\frac{\partial F}{\partial b}(0,b) = -\lambda_1$. This is consistent with the implicit derivative in (4.10) since both statements demand that the ratio b^2/a^2 converges to $(\lambda_2/\lambda_1)^2$ as (a,b) approaches the origin along Γ_1.

By definition Γ_1 must be the graph of B_F, which is continuously differentiable with negative slope whereves it lies inside $(-1,1)^2$ and has slope zero when $|a| = 1$. Given that $0 < B_F(-1) < B_F(1) < 1$ and $0 < A_F(-1) < A_F(1) < 1$, it follows that Γ_1 consists of two separate arcs. One arc, denoted by Γ_1^1, connects the points $(A_F(1), 1)$ and $(1, B_F(1))$ within the first quadrant. The second arc, denoted by Γ_1^2, connects $(-1, B_F(-1))$ to $(A_F(-1), -1)$ and passes through the origin (since $F(0,0) = 0$) along the way.

Finally, a routine derivation shows that

$$\left.\frac{d^2 b}{da^2}\right|_{\Gamma_1} = \frac{2\lambda_1}{\lambda_2^2}\frac{b^2}{a^4(1-b^2)^3}\left(\lambda_1 b(1-a^2)^2 + \lambda_2 a(1-b^2)^2\right), \tag{4.15}$$

which is clearly positive if $a, b > 0$, thus proving that Γ_1^1 is convex. Using again that $F(a,b) = 0$, we rewrite the second derivative in the form

$$\begin{aligned}\frac{d^2 b}{da^2} &= \frac{2\lambda_1}{\lambda_2^2}\frac{b^2}{a^4(1-b^2)^3}\left(F(a,b) + \lambda_1 b(1-a^2)^2 + \lambda_2 a(1-b^2)^2\right) \\ &= \frac{2\lambda_1}{\lambda_2^2}\frac{b^2}{a^4(1-b^2)^3}\,ab\left(\omega^2 + \lambda_1(1-a)^2(2+a) + \lambda_2(1-b)^2(2+b)\right)\end{aligned}$$

So long as $0 < |a|, |b| < 1$, the sign of this second derivative is determined by the sign of ab, which is negative everywhere on Γ_1^2 except the origin. A separate calculation shows that

$$\frac{d^2b}{da^2}(0,0) = \frac{-2\lambda_2(\omega^2 + 2\lambda_1 + 2\lambda_2)}{\lambda_1^2} < 0, \tag{4.16}$$

finishing the proof. The above expression can be simplified further by noting that $\omega^2+2\lambda_1+2\lambda_2 = \gamma^2$ when $a = b = 0$. □

Remark 4.3. Recalling that $a = \cos k_1$ and $b = \cos k_2$ for points $(k_1, k_2) \in [-\pi, \pi]^2$, we can reconstruct the graph of Φ_1 (defined in (4.5)) from the graph of Γ_1 (see Figure 6). The arc Γ_1^1 corresponds to the closed curve around zero. The origin in the ab-plane has the four points $(\pm\pi/2, \pm\pi/2)$ as its inverse image and the arc Γ_1^2 turns into the closed curve around the point (π, π) on the compactified torus $[-\pi, \pi]^2$.

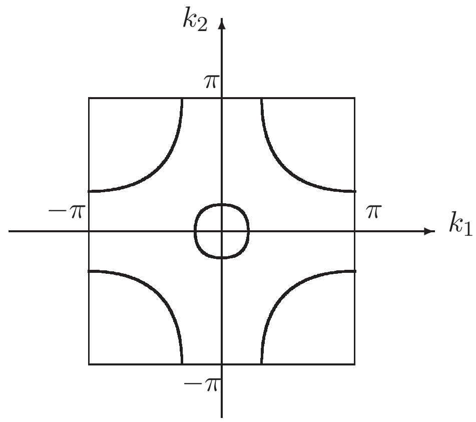

FIGURE 6. Set Φ_1

By definition a function $\phi_v(k) = k \cdot v - \gamma(k)$ may have a degenerate critical point only at $k^* \in \Phi_1$, and this occurs with the choice $v = \nabla\gamma(k^*)$. For each $k \in \Phi_1$, let $\xi = \xi(k)$ be an eigenvector of $D^2\gamma(k)$ corresponding to its zero eigenvalue (this is unique up to scalar mutiplication by Proposition 4.1). It follows that $\partial_\xi^2\gamma(k) = 0$, where the notation $\partial_\xi = \xi \cdot \nabla$ indicates a directional derivative as in Section 3. According to the partition (2.6), points $k \in \Phi_1$ belong to either K_2 or K_3 depending on whether the third derivative of γ in the direction of ξ is also zero. The following result is helpful.

Lemma 4.4. *Let $U \subset \mathbb{R}^d$ be a neighborhood of a point k_0 and let $f \in C^3(U)$. Assume that the Hessian matrix D^2f has a zero eigenvalue of multiplicity one at k_0 and let ξ be a corresponding eigenvector. Then*

$$\partial_\xi^3 f(k_0) = 0 \tag{4.17}$$

if and only if

$$\partial_\xi(\det D^2 f)(k_0) = 0. \tag{4.18}$$

Proof. Apply a unitary change of variables to change the coordinate system to one that diagonalizes the matrix $(D^2f)(k_0)$ and in which ξ points in the direction of e_1. In the new system, the only non-zero term of the gradient of $\det(D^2f)(k)$ at k_0 is the gradient of $(\partial_{11}^2 f)(k)$ at k_0 multiplied by a nonzero scalar - the product of all nonzero eigenvalues of $(D^2f)(k_0)$. On the other hand, we have that

$$(\partial_{11}^2 f)(k) = \frac{1}{\|\xi\|^2}\xi^T D^2 f(k)\xi = \frac{1}{\|\xi\|^2}\partial_\xi^2 f(k), \tag{4.19}$$

showing that $(\nabla(\det D^2 f))(k_0)$ is a non-zero scalar multiple of $\nabla(\partial_\xi^2 f)(k_0)$. $\square$

Lemma 4.4 allows us to determine whether points $k \in \Phi_1$ satisfy $\partial_\xi^3 \gamma(k) = 0$ by identifying the set of solutions of

$$\partial_\xi(\det D^2\gamma)(k) = 0. \tag{4.20}$$

Using equations (4.4), (4.7), and notation (4.1) we have

$$\nabla \det D^2\gamma(k)\big|_{k\in\Phi_1} = \frac{\lambda_1\lambda_2}{\gamma^4(k)}\nabla F(k) = -\frac{\lambda_1\lambda_2}{\gamma^4(k)}\left(\partial_a F \sin k_1,\ \partial_b F \sin k_2\right). \tag{4.21}$$

The components of ξ can be constructed from the elements of the matrix (4.9), with one possible choice being

$$\xi(k) = \begin{pmatrix} \partial_a F \\ \lambda_1 \sin k_1 \sin k_2 \end{pmatrix}. \tag{4.22}$$

Note that $\xi(k)$ vanishes as $\sin k_1$ approaches zero along Φ_1 (for example by applying (4.13)), and in fact

$$\frac{1}{\sin k_1}\xi(k) \to \begin{pmatrix} 0 \\ \lambda_1 \sin k_2 \end{pmatrix} \neq 0 \tag{4.23}$$

when $k_1 \to 0, \pm\pi$ along this curve. The combination of (4.22) with (4.21) yields

$$\partial_\xi(\det D^2\gamma)(k) = -\frac{\lambda_1\lambda_2 \sin k_1}{\gamma^4(k)}\left((\partial_a F)^2 + \lambda_1(1-b^2)\partial_b F\right), \quad k \in \Phi_1. \tag{4.24}$$

One should not be concerned with the vanishing of $\sin k_1$ in this formula as it can be counteracted by modifying (4.22) by a suitable scalar multiple. Vanishing of the second factor determines whether $k \in \Phi_1$ belongs to K_2 or K_3. Using (4.12) and (4.13), we have

$$a^2 b\left((\partial_a F)^2 + \lambda_1(1-b^2)\partial_b F\right)\Big|_{\Gamma_1} = \lambda_1^2 b^3(1-a^2)^2 + \lambda_1\lambda_2 a^3(1-b^2)^2$$

and thus if $k \in \Phi_1$ (equivalently if $(a,b) \in \Gamma_1$), then (4.20) holds only if

$$\tilde{G}(a,b) = \lambda_1 b^3(1-a^2)^2 + \lambda_2 a^3(1-b^2)^2 = 0. \tag{4.25}$$

The function $\tilde{G}$ is symmetric in a and b and is rather elegant, but it turns out not to be ideal for our purposes. Restricting our view to $k \in \Phi_1$, we are again free to add any multiple of F to $\tilde{G}$ and work with that object instead. We therefore introduce

$$\begin{aligned} G(a,b) &= \tilde{G}(a,b) + a^2b^2F(a,b) \\ &= \omega^2 a^3 b^3 + \lambda_1 b^3(1-3a^2+2a^3) + \lambda_2 a^3(1-3b^2+2b^3). \end{aligned} \tag{4.26}$$

The function G maintains the property that among all $k \in \Phi_1$, (4.20) holds only if $G(a,b) = 0$. We will describe some features of the set

$$\Gamma_2 = \{(a,b) \in [-1,1]^2 : G(a,b) = 0\} \tag{4.27}$$

as an independent object before seeking out its intersection with Γ_1. The following result is an analog of Lemma 4.2 and we provide it for the sake of completeness.

Lemma 4.5. *The set Γ_2 defined in* (4.27) *is nonempty and is of the form displayed in Figure 7: it consists of two continuous arcs, $\Gamma_2 = \Gamma_2^1 \cup \Gamma_2^2$. The equation $G(a,b) = 0$ defines an implicit function in $[-1,1]^2$ that is locally continuously differentiable with respect to a at any $(a,b) \in \Gamma_2$, $|b| \neq 1$. In particular, the arc Γ_2^2 represents the graph of a function which we denote by $b = B_G(a)$. The following properties hold:*

(1) For any $(a,b) \in \Gamma_2^2 \backslash \{(0,0)\}$, $|b| \neq 1$,

$$\frac{dB_G}{da}(a,b) = -\frac{\lambda_1}{\lambda_2}\frac{b^4(1-a^2)}{a^4(1-b^2)} \leq 0, \tag{4.28}$$

and

$$\frac{dB_G}{da}(0,0) = -\left(\frac{\lambda_2}{\lambda_1}\right)^{1/3}. \tag{4.29}$$

(2) The first arc, Γ_2^1, passes through the third quadrant. The second arc, Γ_2^2, is located in the second and fourth quadrant.

The arc Γ_2^2 is also the graph of a function $a = A_G(b)$. All of the statements (or their analogs) in this lemma remain true if (a, A_G, λ_1) and (b, B_G, λ_2) switch roles.

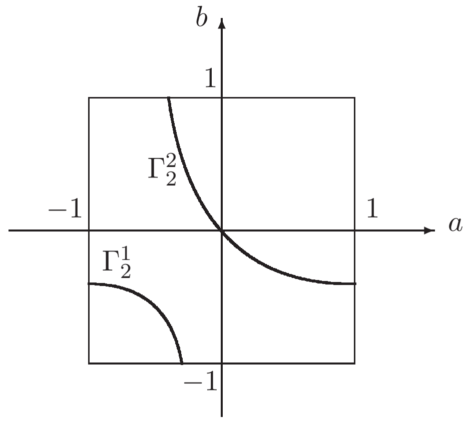

FIGURE 7. Set Γ_2

Proof. The proof is largely identical to that of Lemma 4.2 and we omit the common details. One difference is that a slope at the origin cannot be determined from the ratio of $\frac{\partial G}{\partial a}(0,0)$ and $\frac{\partial G}{\partial b}(0,0)$, as both quantities are zero already.

Note that $G(0,b) = \lambda_1 b^3$ is zero only if $b = 0$. When $a \neq 0$, write $r(a) = \frac{b}{a}$ to obtain the expression

$$G(a,r) = a^3\big((\lambda_1 r^3 + \lambda_2) + a^2(-3\lambda_1 r^3 - 3\lambda_2 r^2) + a^3(\omega^2 + 2\lambda_1 + 2\lambda_2)r^3\big). \tag{4.30}$$

For a fixed value of a, the solutions of $G(a,r) = 0$ occur at the roots of a cubic polynomial whose coefficients depend smoothly on a. When $a = 0$ the polynomial is $\lambda_1 r^3 + \lambda_2$, which has a single transversal root at $r = -(\frac{\lambda_2}{\lambda_1})^{1/3}$. The implicit function theorem provides a neighborhood of $a = 0$ and a continuous function $r(a)$ along which $G(a, r(a)) = 0$.

By definition $\frac{dB_G}{da}(0,0) = \lim_{a\to 0} r(a) = -(\frac{\lambda_2}{\lambda_1})^{1/3}$. Once again the result is consistent with the general implicit derivative (4.28) because the ratio (b^4/a^4) converges to $(\lambda_2/\lambda_1)^{4/3}$ as (a,b) approaches the origin along Γ_2.

□

We are most interested in the intersection points of Γ_1 and Γ_2, which are described in the following result.

Lemma 4.6. *Let the curves Γ_1 and Γ_2 be defined by (4.8) and (4.27), respectively. Then*

(i) *if $\lambda_1 < \lambda_2$, $\Gamma_1 \cap \Gamma_2 = \{(0,0), (a^*, b^*)\}$, with $a^* < 0 < b^*$,*
(ii) *if $\lambda_1 > \lambda_2$, $\Gamma_1 \cap \Gamma_2 = \{(0,0), (a^*, b^*)\}$, with $b^* < 0 < a^*$,*

(iii) *if* $\lambda_1 = \lambda_2$, $\Gamma_1 \cap \Gamma_2 = \{(0,0)\}$.

In the last case, we will use the notation $(a^*, b^*) = (0,0)$.

Proof. Consider the case $\lambda_1 < \lambda_2$. The proof in the case $\lambda_1 > \lambda_2$ is identical.

Note that since the origin belongs to both Γ_1 and Γ_2, it is obviously in their intersection. Lemma 4.2 and Lemma 4.5 show that $\Gamma_2^1 \cap \Gamma_1 = \Gamma_1^1 \cap \Gamma_2 = \emptyset$ and thus, $\Gamma_1 \cap \Gamma_2 = \Gamma_2^2 \cap \Gamma_1^2$. Next, according to Lemma 4.2, Γ_1^2 is concave, and with the assumption $\lambda_2 > \lambda_1$ its slope at zero is less than -1 (see equation (4.11)). As a result, for all $(a,b) \in \Gamma_1^2$ in the fourth quadrant, $|b| > |a|$, except for the origin. Define

$$a^+ = \max\{a : (a,b) \in \Gamma_1 \cap \Gamma_2\}. \tag{4.31}$$

If we assume that Γ_1^2 intersects Γ_2^2 in the fourth quadrant away from the origin, we have that $0 < a^+ < \tilde{a}$, where $\tilde{a} = A_F(-1)$. Comparing formulas (4.10) and (4.28) gives

$$\frac{dB_G}{da}(a) = \frac{dB_F}{da}(a)\frac{b^2}{a^2}, \qquad \text{for all } (a,b) \in \Gamma_1 \cap \Gamma_2,\ (a,b) \neq (0,0). \tag{4.32}$$

Both derivatives are negative, and furthermore $|b| > |a|$ along this part of the curve Γ_1^2. Consequently

$$\frac{dB_G}{da}(a^+) < \frac{dB_F}{da}(a^+), \tag{4.33}$$

and thus, $B_G(a) < B_F(a)$ for all $a \in (a^+, a^+ + \epsilon)$ for some small $\epsilon > 0$. On the other hand, it follows from the results of Lemma 4.5 that $B_G(\tilde{a}) > -1 = B_F(\tilde{a})$, implying that there must exist another point $a' \in [a^+ + \epsilon, \tilde{a})$ such that $B_G(a') = B_F(a')$. This contradicts the definition of a^+ and we may conclude that $\Gamma_1 \cap \Gamma_2 = \{(0,0)\}$ in the fourth quadrant.

We claim that Γ_1^2 intersects Γ_2^2 exactly once in the second quadrant away from the origin. First, since

$$\frac{dB_F}{da}(0) < \frac{dB_G}{da}(0), \tag{4.34}$$

there is a small $\epsilon > 0$ so that $B_F(a) > B_G(a)$ for all $a \in (-\epsilon, 0)$. However for the value $\hat{a} = A_G(1)$, one has $B_F(\hat{a}) < 1 = B_G(\hat{a})$. The Intermediate Value Theorem implies the existence of $a' \in (\hat{a}, 0)$ such that $B_F(a') = B_G(a')$, giving rise to at least one non-origin point of intersection of Γ_1^2 and Γ_2^2 in the second quadrant.

Note that $G(a,-a) = -(\omega^2 + 4\lambda_1)a^6 + (\lambda_2 - \lambda_1)a^3(1 - 3a^2 - 2a^3) < 0$ over the interval $a \in [-1, 0)$. Together with (4.29) that implies that Γ_2^2 lies above the line $b = -a$ within the second quadrant, and it follows from (4.32) that $0 > \frac{dB_F}{da}(a) > \frac{dB_G}{da}(a)$ whenever Γ_1^2 and Γ_2^2 intersect with $a < 0$. On the other hand, if there were multiple points of intersection, the orientation of crossing would have alternating signs. One concludes that $\Gamma_1^2 \cap \Gamma_2^2$ contains a single point (a^*, b^*) in the second quadrant along with the origin.

Now consider the case $\lambda_1 = \lambda_2 = \lambda$, and let $(a_0, b_0) \in \Gamma_1 \cap \Gamma_2$. Then $\tilde{G}(a_0, b_0) = 0$, where $\tilde{G}$ is defined in (4.25). Since λ_1 and λ_2 are equal, $\tilde{G}$ admits the factorization

$$\tilde{G}(a,b) = \lambda(a+b)(a^2 + b^2 + ab(a^2b^2 - 2ab - 1)). \tag{4.35}$$

The second factor is zero if and only if $|a| = |b| = 1$, however those points do not belong to Γ_1. Hence, $b_0 = -a_0$. Plugging this into (4.7), one can see that when $\lambda_1 = \lambda_2$, $F(a_0, -a_0) = 0$ if and only if $a_0 = 0$. The intersection of Γ_1 and Γ_2 at the origin is not transversal in this case, but instead the two curves are tangent without crossing. $\square$

Remark 4.7. Introduce the set K^* as the pre-image of the point (a^*, b^*) under the map (4.1). The set K^* consists of four points on the set Φ_1 that are located as shown on Figure 8. Note that in the case $\lambda_1 = \lambda_2$, $K^* = \left\{(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})\right\}$.

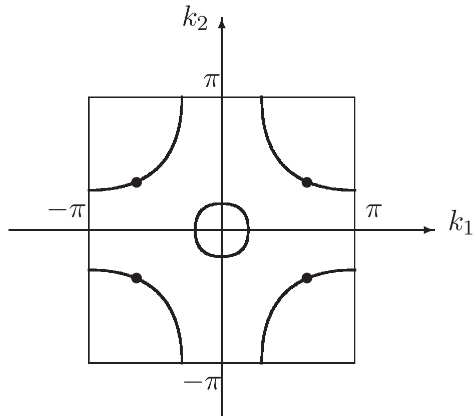

Figure 8. Set Φ_1 along with the four-point set K^*

The following is an easy consequence of Lemma 4.6.

Corollary 4.8. *Assume that* $\det D^2\gamma(k) = 0$ *and let* $\xi = \xi(k)$ *be an eigenvector of* $D^2\gamma(k)$ *corresponding to the zero eigenvalue. Let* K^* *be the set defined in Remark 4.7. Then* $\partial_\xi^3\gamma(k) = 0$ *if and only if* $k \in K^*$.

Proof. First recall that the assumption $\det D^2\gamma(k) = 0$ is equivalent to the fact that the corresponding $(a, b) \in \Gamma_1$ (see (4.6)).

Assume that $\partial_\xi^3\gamma(k) = 0$. In Lemma 4.4 we showed that under our main assumption this condition is equivalent to (4.20). This, in turn, implies that $G(a, b) = 0$ (or $(a, b) \in \Gamma_2$), where (a, b) is an image of k under the map (4.1). We may therefore conclude that $(a, b) \in \Gamma_1 \cap \Gamma_2$. According to Lemma 4.6, in the case $\lambda_1 = \lambda_2$, $(a, b) = (a^*, b^*)$ and thus, $k \in K^*$. If $\lambda_1 \neq \lambda_2$, (a, b) is either (a^*, b^*) or the origin. However, a direct inspection (see equation (4.24)) shows that when $(a, b) = (0, 0)$,

$$\partial_\xi(\det D^2\gamma)(k) = \pm\frac{\lambda_1\lambda_2}{\gamma^4(k)}\left(\lambda_1^2 - \lambda_2^2\right) \neq 0, \tag{4.36}$$

and we can exclude the origin from consideration. Thus, $(a, b) = (a^*, b^*)$ and $k \in K^*$.

The proof in the reverse direction is similar. Assume $k \in K^*$, then it corresponds to (a^*, b^*). Again, in the case $\lambda_1 = \lambda_2$, $(a^*, b^*) = (0, 0)$, and by (4.36), $\partial_\xi(\det D^2\gamma)(k) = \partial_\xi^3\gamma(k) = 0$. If $\lambda_1 \neq \lambda_2$, both a^* and b^* are different from zero. Comparing equations (4.24) – (4.26), we see that $G(a^*, b^*)$ is a nonzero multiple of $a^*(b^*)^2\partial_\xi(\det D^2\gamma)(k)$. Since $G(a^*, b^*) = 0$ and $a^*(b^*)^2 \neq 0$, we conclude $\partial_\xi(\det D^2\gamma)(k) = \partial_\xi^3\gamma(k) = 0$.

□

Lemma 4.9. *Let* γ *be defined by (4.2) and let* $k^* \in K^*$*, where* K^* *is defined in Remark 4.7. Furthermore, let* $\xi = \xi(k^*)$ *be the eigenvector of the Hessian matrix of* γ *at* k^* *corresponding to the zero eigenvalue and* $\xi^\perp$ *be a vector orthogonal to* ξ *and of the same magnitude. Then*

$$\left(\partial_\xi^4\gamma\partial_{\xi^\perp}^2\gamma - 3\left(\partial_\xi^2\partial_{\xi^\perp}\gamma\right)^2\right)(k^*) \neq 0. \tag{4.37}$$

Proof. To prove equation (4.37), it is enough to show that

$$\left(\partial_\xi^4\gamma\partial_{\xi^\perp}^2\gamma - 2\left(\partial_\xi^2\partial_{\xi^\perp}\gamma\right)^2\right)(k^*) < 0. \tag{4.38}$$

The above expression admits the following short representation that we will use in our calculations,

$$\left(\partial_\xi^4\gamma\partial_{\xi^\perp}^2\gamma - 2\left(\partial_\xi^2\partial_{\xi^\perp}\gamma\right)^2\right)(k^*) = \|\xi\|^4\left(\partial_\xi^2\det D^2\gamma\right)(k^*). \tag{4.39}$$

Let us first prove (4.39). Indeed, we can re-write $\det D^2\gamma$ in the new coordinates as

$$\|\xi\|^4 \det D^2\gamma = \partial_\xi^2\gamma\partial_{\xi^\perp}^2\gamma - \left(\partial_\xi\partial_{\xi^\perp}\gamma\right)^2. \tag{4.40}$$

Differentiating the above equation we obtain

$$\|\xi\|^4\partial_\xi^2 \det D^2\gamma = \partial_\xi^4\gamma\partial_{\xi^\perp}^2\gamma + 2\partial_\xi^3\gamma\partial_\xi\partial_{\xi^\perp}^2\gamma + \partial_\xi^2\gamma\partial_\xi^2\partial_{\xi^\perp}^2\gamma - 2\left(\partial_\xi^2\partial_{\xi^\perp}\gamma\right)^2 - 2\partial_\xi\partial_{\xi^\perp}\gamma\partial_\xi^3\partial_{\xi^\perp}\gamma.$$

At the point k^*, the quantities $\partial_\xi^2\gamma$, $\partial_\xi\partial_{\xi^\perp}\gamma$, and $\partial_\xi^3\gamma$ all vanish, thus the second, third and fifth term of the above equation vanish as well, proving (4.39).

Next, it is easy to see that

$$\left(\partial_\xi^2 \det D^2\gamma\right)(k^*) = \left(\partial_\xi^2 F\right)(k^*)\frac{\lambda_1\lambda_2}{\gamma^4(k^*)}, \tag{4.41}$$

where F is defined in (4.7). Finally, a direct calculation shows that in the case $\lambda_1 \neq \lambda_2$,

$$\left(\partial_\xi^2 F\right)(k^*) = 2\|\xi\|^2\gamma^2(k^*)\frac{a^*b^*(1-(a^*)^2)(1-(b^*)^2)}{(a^*)^2(1-(b^*)^2)+(b^*)^2(1-(a^*)^2)} < 0, \tag{4.42}$$

and in the case $\lambda_1 = \lambda_2 = \lambda$,

$$\left(\partial_\xi^2 F\right)(k^*) = -\|\xi\|^2(\omega^2 + 2(\lambda_1+\lambda_2)) < 0, \tag{4.43}$$

finishing the proof. □

4.1. **Proof of Proposition 2.2.** The curves of Φ_1 consist of points where $\det D^2\gamma(k) = 0$, which are also the points where the "velocity map" $\mathcal{V}(k) = \nabla\gamma(k)$ does not satisfy the hypotheses of the inverse function theorem. As a result the boundary of $\mathcal{V}([-\pi,\pi]^2)$ must be a subset of $\Psi_1 \cup \Psi_2 = V(\Phi_1)$ as defined in Proposition 2.2.

Recall from Remark 4.3 that Φ_1 has one closed curve around zero and a second closed curve around the point (π,π). Let Ψ_1 be the image of the former under the velocity map and let Ψ_2 be the image of the latter. The analysis of Ψ_1 is more straightforward because the points of K^* are not involved.

In vector form, the velocity map is

$$\nabla\gamma(k) = \frac{1}{\gamma(k)}\begin{pmatrix}\lambda_1 \sin k_1\\ \lambda_2\sin k_2\end{pmatrix}.$$

Thus points k in a given "quadrant" of the torus $[-\pi,\pi]^2$ are mapped to the same quadrant of $\mathbb{R}^2$.

The tangent line to Φ_1 always points in the direction normal to $\nabla\det D^2\gamma(k)$, which by (4.21) is also normal to $\nabla F(k)$. Suppose k travels along Φ_1 with instantaneous velocity $\begin{pmatrix}-\partial_b F\sin k_2\\ \partial_a F\sin k_1\end{pmatrix}$. Then k follows either loop of Φ_1 through the four quadrants of the compactified torus in order, and $\mathcal{V}(k)$ must wind once around the origin.

At a local level, the differential $d\mathcal{V}(k)$ is the Hessian matrix $D^2\gamma(k)$, whose image is spanned by the direction $\xi^\perp$. Plugging (4.9) into the Leibniz rule we determine that $\mathcal{V}(k)$ moves with velocity

$$\begin{aligned}
&\frac{1}{\gamma^3(k)}\begin{pmatrix}\lambda_1\partial_b F & -\lambda_1\lambda_2\sin k_1\sin k_2\\ -\lambda_1\lambda_2\sin k_1\sin k_2 & \lambda_2\partial_a F\end{pmatrix}\begin{pmatrix}-\partial_b F\sin k_2\\ \partial_a F\sin k_1\end{pmatrix}\\
&\qquad= \frac{1}{\gamma^3(k)}\begin{pmatrix}-\lambda_1(\partial_b F)^2 - \lambda_1\lambda_2(1-a^2)\partial_a F\sin k_2\\ \lambda_1\lambda_2(1-b^2)\partial_b F\sin k_1 + \lambda_2(\partial_a F)^2\sin k_2\end{pmatrix}\\
&\qquad= \frac{\lambda_1\lambda_2}{\gamma^3(k)}\left[\frac{\lambda_1 b^3(1-a^2)^2 + \lambda_2 a^3(1-b^2)^2}{a^2 b}\right]\begin{pmatrix}-\frac{a}{b}\sin k_2\\ \sin k_1\end{pmatrix}\\
&\qquad= \frac{\lambda_1\lambda_2}{\gamma^3(k)}\left[\frac{\tilde{G}(a,b)}{a^2 b}\right]\begin{pmatrix}-\frac{a}{b}\sin k_2\\ \sin k_1\end{pmatrix}.
\end{aligned} \tag{4.44}$$

Identities (4.12) and (4.13) are used multiple times between the second and third lines.

The prefactor of $\lambda_1\lambda_2\gamma^{-3}(k)$ is positive for all k. The factor of $\tilde{G}/(a^2b)$ is strictly positive as k traces out the loop of Φ_1 circling the origin because $a, b > 0$ here. The discussion leading up to Lemma 4.6 and Corollary 4.8 shows that for general $k \in \Phi_1$, the value of $\tilde{G}/(a^2b)$ changes sign when k crosses a point of K_3 and at no other time.

The vector with components $(-\frac{a}{b}\sin k_2, \sin k_1)$ points in the direction of $\xi^\perp$ and does not vanish while $k \in \Phi_1$ (the points where $a = b = 0$ are handled by (4.11)). Indeed one could choose this vector as the definition of $\xi^\perp(k)$. Consider $\xi^\perp$ as measured in polar coordinates. The path of $\mathcal{V}(k)$ turns to the left or the right depending on whether the polar angle of $\xi^\perp$ is increasing or decreasing with k. The direction of change for this angle in turn depends on the sign of the determinant

$$\det\begin{pmatrix} -\frac{a}{b}\sin k_2 & -\big(\lambda_2\frac{a(1-b^2)^2}{b^2} + \lambda_1\frac{1-a^2}{b}\big)\sin k_1 \\ \sin k_1 & -\lambda_2\frac{a^2(1-b^2)}{b}\sin k_2 \end{pmatrix} = \frac{1}{b^2}\big(\lambda_2 a(1-b^2)^2 + \lambda_1 b(1-a^2)^2\big).$$

The left column of the 2×2 matrix above is $\xi^\perp(k)$. The right column is its rate of change as k travels along Φ_1 at the prescribed velocity, computed via the product $\begin{pmatrix} \frac{\sin k_1 \sin k_2}{b} & -\frac{a}{b^2} \\ a & 0 \end{pmatrix}\begin{pmatrix} -\partial_b F \sin k_2 \\ \partial_a F \sin k_1 \end{pmatrix}$.

In the analysis of (4.15), this quantity is shown to have the same sign as ab, which is positive on the loop of Φ_1 corresponding to Ψ_1 and negative on the loop corresponding to Ψ_2.

One last note is that the sign of $\frac{a}{b}$ is constant on the connected components of Φ_1, so the direction of $\xi^\perp$ winds exactly once around the origin on each loop. Combined with the preceding facts, it follows that Ψ_1 is a simple closed curve enclosing a convex region in $\mathbb{R}^2$, and that Ψ_2 is a simple closed curve composed of four arcs with the opposite curvature from Ψ_1. These arcs intersect at cusps located at the points V_3 corresponding to values of $k \in K_3$ where $\tilde{G}/(a^2b)$ changes sign.

It remains to be shown that Ψ_1 is the boundary of $\mathcal{V}([-\pi,\pi]^2)$ and that Ψ_1, Ψ_2 are disjoint. By comparing supplementary angles, it is clear that the extreme values of $\mathcal{V}(k)$ in any given direction must occur while $|k_1|, |k_2| \le \frac{\pi}{2}$. With the exception of $k = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$, all points where $\mathcal{V}(k) \in \Psi_2$ satisfy $\cos k_1 \cos k_2 < 0$, so one of the coordinates is necessarily greater than $\frac{\pi}{2}$. When $|k_1| = |k_2| = \frac{\pi}{2}$, the vector $\xi^\perp$ which spans the image of $D\mathcal{V}$ happens to be collinear with $\mathcal{V}(k)$, so these choices for k do not generate extreme points of $\mathcal{V}([-\pi,\pi]^2)$ in their respective directions. By process of elimination, the boundary must consist of Ψ_1 alone.

5. Applications to quantum systems

In this section we apply the main integral estimates from Section 2 to obtain dispersive estimates in a class of infinite-volume harmonic systems. The material of the two introductory subsections follow [1], we provide it here for completeness. For more details, see [1] and references therein.

5.1. **Harmonic evolutions in finite volume.**

We first introduce a class of finite volume harmonic systems on two-dimensional square lattices. For an integer $L \ge 1$, denote $\Lambda_L = (-L, L]^2 \subset \mathbb{Z}^2$. We associate with each $x \in \Lambda_L$ the position and momentum operators on $L^2(\mathbb{R}, dq_x)$: $\hat{q}_x$ is the operator of multiplication by q_x and $\hat{p}_x = -i\frac{d}{dq_x}$.

We denote by q_x and p_x the extensions of the corresponding single-site operators to the full Hilbert space

$$\mathcal{H}_L = \bigotimes_{x\in\Lambda_L} L^2(\mathbb{R}, dq_x), \tag{5.1}$$

defined by setting

$$q_x = \mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\hat{q}_x\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1} \quad\text{and}\quad p_x = \mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes -i\frac{d}{dq_x}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}. \tag{5.2}$$

Operators q_x and p_x are self-adjoint on $\mathcal{H}_L$ and satisfy the canonical commutation relations (CCR):

$$[p_x, p_y] = [q_x, q_y] = 0 \quad \text{and} \quad [q_x, p_y] = i\delta_{x,y}\mathbb{1}\,, \tag{5.3}$$

for all $x, y \in \Lambda_L$.

The Hamiltonian

$$H_L = \frac{1}{2}\sum_{x\in\Lambda_L} p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^{2} \lambda_j \,(q_x - q_{x+e_j})^2 \tag{5.4}$$

is a self-adjoint operator on Hilbert space $\mathcal{H}_L$ and represents a system of coupled harmonic oscillators. In the above, $\{e_j\}_{j=1}^2$ represents the canonical basis vectors in $\mathbb{Z}^2$, the parameters $\lambda_j, \omega > 0$ represent the coupling strength and the on-site energy. Finally, the Hamiltonian H_L is equipped with periodic boundary conditions, i.e., if $x \in \Lambda_L$ but $x + e_j \notin \Lambda_L$, set $q_{x+e_j} = q_{x-(2L-1)e_j}$.

Let $\mathcal{B}(\mathcal{H}_L)$ denote the space of bounded linear operators over the Hilbert space $\mathcal{H}_L$. We will refer to elements of $\mathcal{B}(\mathcal{H}_L)$ as observables. As a self-adjoint operator, the Hamiltonian H_L generates the time evolution τ_t^L on $\mathcal{B}(\mathcal{H}_L)$, given by

$$\tau_t^L(A) = e^{itH_L} A e^{-itH_L} \quad \text{for all } t \in \mathbb{R} \quad \text{and all } A \in \mathcal{B}(\mathcal{H}_L)\,. \tag{5.5}$$

τ_t^L is a well-defined, one parameter group of $*$-automorphisms, called the Heisenberg dynamics.

A special class of observables called Weyl operators are important for the transition to the infinite-volume systems. For any $f : \Lambda_L \to \mathbb{C}$, a unitary operator

$$W(f) = \exp\left[i \sum_{x\in\Lambda_L} (\mathrm{Re}[f(x)]q_x + \mathrm{Im}[f(x)]p_x) \right] \tag{5.6}$$

is called a Weyl operator or a Weyl observable. One can show that

$$W(f)^* = W(-f) \tag{5.7}$$

and

$$W(f)W(g) = e^{-i\mathrm{Im}[\langle f,g\rangle]/2} W(f+g) \tag{5.8}$$

hold for all $f, g : \Lambda_L \to \mathbb{C}$. In addition, $W(0) = \mathbb{1}$ and $\|W(f) - \mathbb{1}\| = 2$ for each $f \neq 0$. An important fact is that the set of Weyl operators is invariant with respect to τ_t^L: there exists a family of operators $T_t^L : \ell^2(\Lambda_L) \to \ell^2(\Lambda_L)$ such that

$$\tau_t^L(W(f)) = W(T_t^L f)\,. \tag{5.9}$$

Formula (5.9) is verified by diagonalizing H_L with Fourier-type operators (cf. [13]). The operators T_t^L have an explicit construction

$$T_t^L = (U + V)\mathcal{F}^{-1} M_t \mathcal{F}(U^* - V^*)\,. \tag{5.10}$$

Here $\mathcal{F} : \ell^2(\Lambda_L) \to \ell^2(\Lambda_L^*)$ is the unitary Fourier transform with $\Lambda_L^* = \left\{ \dfrac{x\pi}{L} : x \in \Lambda_L \right\}$ and M_t is the operator of multiplication by $e^{2i\gamma t}$, where

$$\gamma(k) = \sqrt{\omega^2 + 4\sum_{j=1}^{2} \lambda_j \, \sin^2(k_j/2)}\,, \quad k \in \Lambda_L^*. \tag{5.11}$$

Operators U and V, known as Bogoliubov transformations (cf. [9]), are given by

$$U = \frac{i}{2}\mathcal{F}^{-1} M_{\Gamma_+} \mathcal{F} \quad \text{and} \quad V = \frac{i}{2}\mathcal{F}^{-1} M_{\Gamma_-} \mathcal{F} J, \tag{5.12}$$

where J is complex conjugation, and $M_{\Gamma_\pm}$ is the operator of multiplication by

$$\Gamma_\pm(k) = \frac{1}{\sqrt{\gamma(k)}} \pm \sqrt{\gamma(k)}\,, \tag{5.13}$$

with $\gamma(k)$ as in (5.11).

5.2. **Harmonic evolutions in infinite volume.** We start our discussion of infinite harmonic lattice with a review of general Weyl algebra formalism (see [10] and [2] for details).

Let $\mathcal{D}$ denote a real linear space equipped with a symplectic, non-degenerate bilinear form σ. The Weyl algebra over $\mathcal{D}$, which we will denote by $\mathcal{W}(\mathcal{D})$, is then defined to be a C^*-algebra generated by Weyl operators, i.e., non-zero elements $W(f)$, associated to each $f \in \mathcal{D}$, which satisfy

$$W(f)^* = W(-f) \quad \text{for each } f \in \mathcal{D}\,, \tag{5.14}$$

and

$$W(f)W(g) = e^{-i\sigma(f,g)/2}W(f+g) \quad \text{for all } f, g \in \mathcal{D}\,. \tag{5.15}$$

It is well known (cf. [2], Theorem 5.2.8.) that such an algebra with additional properties that $W(0) = \mathbb{1}$, $W(f)$ is unitary for all $f \in \mathcal{D}$, and $\|W(f) - \mathbb{1}\| = 2$ for all $f \in \mathcal{D} \setminus \{0\}$ is unique up to a $*$-isomorphism.

In our case a convenient choice of $\mathcal{D}$ will be $\mathcal{D} = \ell^2(\mathbb{Z}^2)$ or $\mathcal{D} = \ell^1(\mathbb{Z}^2)$ with the symplectic form

$$\sigma(f, g) = \operatorname{Im}\left[\langle f, g\rangle\right] \quad \text{ for } f, g \in \mathcal{D}. \tag{5.16}$$

We denote the corresponding Weyl algebra by $\mathcal{W}(\ell^2(\mathbb{Z}^2))$ or $\mathcal{W}(\ell^1(\mathbb{Z}^2))$.

Let us return to the general case. Another result of Theorem 5.2.8 of [2] is that a group of real linear mappings $\{T_t\}_{t\in\mathbb{R}}$, $T_t : \mathcal{D} \to \mathcal{D}$ such that

$$\sigma(T_t f, T_t g) = \sigma(f, g)\,, \quad \text{ for all } t \in \mathbb{R}, \tag{5.17}$$

generates a unique one-parameter group of $*$-automorphisms τ_t on $\mathcal{W}(\mathcal{D})$, such that

$$\tau_t(W(f)) = W(T_t f) \quad \text{for all } f \in \mathcal{D}. \tag{5.18}$$

In order to use this result to introduce a harmonic dynamics on $\mathcal{W}(\ell^2(\mathbb{Z}^2))$, define T_t on $\ell^2(\mathbb{Z}^2)$ by (compare with (5.10))

$$T_t = (U + V)\mathcal{F}^{-1}M_t\mathcal{F}(U^* - V^*)\,. \tag{5.19}$$

Here $\mathcal{F} : \ell^2(\mathbb{Z}^2) \to L^2([-\pi, \pi]^2)$ is the unitary Fourier transform and M_t is the operator of multiplication on $L^2([-\pi, \pi]^2)$ again by $e^{2i\gamma t}$, where the function γ is defined as in (2.1), on $[-\pi, \pi]^2$ (compare to (5.11)). Operators U and V are also defined as in (5.12) with the appropriately extended objects.

It is easy to see that $\{T_t\}_{t\in\mathbb{R}}$ is a family of real linear mappings, that satisfies the group properties $T_0 = \mathbb{1}$, $T_{s+t} = T_s \circ T_t$. Moreover, for each fixed t, T_t is symplectic, i.e.,

$$\operatorname{Im}\left[\langle T_t f, T_t g\rangle\right] = \operatorname{Im}\left[\langle f, g\rangle\right]\,, \tag{5.20}$$

therefore (5.17) is satisfied (see [1] for details). Thus we may conclude that there exists of a unique one-parameter group of $*$-automorphisms on $\mathcal{W}(\ell^2(\mathbb{Z}^2))$, denoted by τ_t, such that

$$\tau_t(W(f)) = W(T_t f) \quad \text{for all } f \in \ell^2(\mathbb{Z}^2)\,. \tag{5.21}$$

The family τ_t is called the infinite volume harmonic dynamics on $\mathcal{W}(\ell^2(\mathbb{Z}^2))$.

5.3. **Main Results.** We start with a standard estimate on the norm of the commutator of two Weyl observables. Using the Weyl relations (5.15), we get

$$\begin{aligned} [\tau_t(W(f)), W(g)] &= \{W(T_t f) - W(g)W(T_t f)W(-g)\} W(g) \\ &= \left\{1 - e^{i\mathrm{Im}[\langle T_t f, g\rangle]}\right\} W(T_t f)W(g) . \end{aligned} \tag{5.22}$$

Since all the Weyl operators are unitary, we have

$$\|[\tau_t(W(f)), W(g)]\| = \left|1 - e^{i\mathrm{Im}[\langle T_t f, g\rangle]}\right| \leq |\langle T_t f, g\rangle| \ , \tag{5.23}$$

for all $f, g \in \ell^2(\mathbb{Z}^2)$. Furthermore, it is easy to see that $T_t f$ has an explicit representation:

$$T_t f = f * \left(H_t^{(0)} - \frac{i}{2}(H_t^{(-1)} + H_t^{(1)})\right) + \overline{f} * \left(\frac{i}{2}(H_t^{(1)} - H_t^{(-1)})\right) \ , \tag{5.24}$$

where

$$\begin{aligned} H_t^{(-1)}(x) &= \frac{1}{(2\pi)^2}\mathrm{Im}\left[\int_{[-\pi,\pi]^2} \frac{1}{\gamma(k)} e^{i(k\cdot x - t\gamma(k))}\, dk\right] , \\ H_t^{(0)}(x) &= \frac{1}{(2\pi)^2}\mathrm{Re}\left[\int_{[-\pi,\pi]^2} e^{i(k\cdot x - t\gamma(k))}\, dk\right] , \\ H_t^{(1)}(x) &= \frac{1}{(2\pi)^2}\mathrm{Im}\left[\int_{[-\pi,\pi]^2} \gamma(k)\, e^{i(k\cdot x - t\gamma(k))}\, dk\right] , \end{aligned} \tag{5.25}$$

(see [12] for the finite volume analog). Combining (5.23) and (5.24), we find that

$$\|[\tau_t(W(f)), W(g)]\| \leq \sum_{x,y} |f(x)|\, |g(y)| \sum_{m \in \{-1,0,1\}} |H_t^{(m)}(x-y)| \, . \tag{5.26}$$

We can now state the results. The first one describes pairs of Weyl operators $W(f)$ and $W(g)$, such that the norm of the commutator $[\tau_t(W(f)), W(g)]$ decays polynomially.

Theorem 5.1. *Fix the parameters $\omega > 0$ and $\lambda_1, \lambda_2 > 0$. Denote by τ_t the harmonic dynamics defined as above on $\mathcal{W}(\ell^2(\mathbb{Z}^2))$. Then there exists a number $C_3 > 0$, such that*

$$\|[\tau_t(W(f)), W(g)]\| \leq \min\left[2, \frac{C_3\|f\|_1\|g\|_1}{|t|^{3/4}}\right] \tag{5.27}$$

holds for all $f, g \in \ell^1(\mathbb{Z}^2)$.

Next, fix a positive number δ and denote $X = \mathrm{supp}(f)$ and $Y = \mathrm{supp}(g)$. Then there exist numbers $C_i = C_i(\delta) > 0$, $i = 1, 2$, such that

$$\|[\tau_t(W(f)), W(g)]\| \leq \min\left[2, \frac{C_2\|f\|_1\|g\|_1}{|t|^{5/6}}\right] \tag{5.28}$$

holds for all $f, g \in \ell^1(\mathbb{Z}^2)$ such that $X - Y \in \mathbb{Z}^2 \setminus B_{t\delta}(tV_3)$, and

$$\|[\tau_t(W(f)), W(g)]\| \leq \min\left[2, \frac{C_1\|f\|_1\|g\|_1}{|t|}\right] \tag{5.29}$$

holds for all $f, g \in \ell^1(\mathbb{Z}^2)$ such that $X - Y \in \mathbb{Z}^2 \setminus B_{t\delta}(t(V_2 \cup V_3))$. Here $B_r(S)$ represents an open neighborhood of radius r of a set S and $X - Y$ is a difference set:

$$X - Y = \{x - y : x \in X, y \in Y\}.$$

Proof. From (5.26), we have

$$\|[\tau_t(W(f)), W(g)]\| \leq \|f\|_1 \|g\|_1 \max_{x,y\in\mathbb{Z}^2} \sum_{m\in\{-1,0,1\}} |H_t^{(m)}(x-y)| . \tag{5.30}$$

By Corollary 2.4, for each $m \in \{-1, 0, 1\}$, there exists a constant $C_m > 0$, such that

$$|H_t^{(m)}(x-y)| \leq \frac{C_m}{|t|^{3/4}}, \qquad \text{for all } |t| \geq 1 \text{ and all } x, y \in \mathbb{Z}^2. \tag{5.31}$$

This proves (5.27) with $C_3 = C_{-1} + C_0 + C_1$.

Next, we show (5.28). Again applying (5.26), we have

$$\|[\tau_t(W(f)), W(g)]\| \leq \|f\|_1 \|g\|_1 \max_{x\in X, y\in Y} \sum_{m\in\{-1,0,1\}} |H_t^{(m)}(x-y)| . \tag{5.32}$$

By the assumption of (5.28), $x-y \notin B_{t\delta}(tV_3)$ for all $x \in X$ and $y \in Y$, therefore each integral in the right-hand side of the above inequality can be estimated by one of the following: (2.11), (2.12), or the result of Theorem 2.7. Thus, for each $m \in \{-1, 0, 1\}$, there exists a constant $C'_m = C'_m(\delta) > 0$, such that

$$|H_t^{(m)}(x-y)| \leq \frac{C'_m}{|t|^{5/6}}, \qquad \text{for all } |t| \geq 1 \text{ and all } x, y \in \mathbb{Z}^2 \text{ such that } x-y \notin B_{t\delta}(tV_3). \tag{5.33}$$

This proves (5.27) with $C_2 = C_2(\delta) = C'_{-1}(\delta) + C'_0(\delta) + C'_1(\delta)$.

The proof of (5.29) is similar. □

The following result is a direct consequence of Theorem 2.6 and Theorem 2.7.

Theorem 5.2. *Fix the parameters $\omega > 0$ and $\lambda_j 1, \lambda_2 > 0$. Denote by τ_t the harmonic dynamics defined as above on $\mathcal{W}(\ell^2(\mathbb{Z}^2))$. Let $X = \operatorname{supp}(f)$ and $Y = \operatorname{supp}(g)$ for $f, g \in \ell^1(\mathbb{Z}^2)$. Then if $X - Y \in tV_0$ with $\operatorname{dist}(X - Y, tV_1) \geq \delta$, then there exist constants $C = C(\delta)$ and $\mu = \mu(\delta)$ such that*

$$\|[\tau_t(W(f)), W(g)]\| \leq C\|f\|_1\|g\|_1 e^{-\mu \operatorname{dist}(X-Y, tV_1)}.$$

Moreover, for every $\mu > 0$ there exist constants $0 < v_\mu \leq \frac{1}{\mu}(1 + 2\sqrt{\lambda_1 + \lambda_2}\sinh(\mu/2))$ and $C_\mu < \omega + 2\sqrt{\lambda_1 + \lambda_2}\cosh(\mu/2)$ such that

$$\|[\tau_t(W(f)), W(g)]\|\, C_\mu \leq e^{-\mu(\operatorname{dist}(X,Y) - v_\mu|t|)}. \tag{5.34}$$

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